Fixed Income Portfolio Design Models

We consider a few basic portfolio design models for fixed income securities. These models are not based on the Markowitz problem or expected utility maximization. Rather they represent general portfolio design problems that attempt to achieve different objectives.

\[ U = \{1, 2, \ldots, I\} \] Denotes the universe of securities
\[ i \in U \] A security from the universe
\[ T = \{1, 2, \ldots, T_{\text{max}}\} \] Denotes a set of discrete points in time
\[ t \in T \] A point in time

\[ \{\} \]
\[ 1, 2, \ldots, \]
\[ UI \]
\[ = \ldots \]
\[ \text{Denotes the universe of securities} \]
\[ i \in U \]
\[ \text{A security from the universe} \]
\[ T = \{1, 2, \ldots, T_{\text{max}}\} \]
\[ \text{Denotes a set of discrete points in time} \]
\[ t \in T \]
\[ \text{A point in time} \]

Copyright F. Novomestky 2004

\[
C_{it} \quad \text{Cash flow for security } i \text{ at time } t \\
x_i \quad \text{Nominal holding of security } i \\
x = \begin{bmatrix} x_1 & x_2 & \cdots & x_I \end{bmatrix} \]
\[ P_i \quad \text{Price or present value for security } i \\
r_i \quad \text{Cash flow yield to maturity for security } i \\
P_i = \sum_{t \in T} \frac{C_{it}}{(1 + r_t)^t} 
\]

Copyright F. Novomestky 2004

\[
\text{Dollar Duration} \\
k_i = \frac{\partial P_i}{\partial r_t} = \sum_{t \in T} \frac{tC_{it}}{(1 + r_t)^t} \\
\]

\[
\text{Portfolio Dollar Duration} \\
k_p = \sum_{i \in U} x_i k_i \\
\]

\[
\text{Present value and duration of liabilities} \\
P_L = k_L 
\]

Copyright F. Novomestky 2004
A common objective function is the portfolio yield which is approximated as follows
\[ r_P \approx \frac{\sum r_i x_i}{\sum k_i x_i} \]

The denominator in the above expression is fixed and is equal to the liability duration. Hence we can write the so called core immunization problem as the following linear programming (LP) problem.

\[
\max \sum_{i=1}^{n} k_i r_i x_i \\
\text{Subject to the following constraints:} \\
\sum_{i=1}^{n} P_i x_i = P_L \\
\sum_{i=1}^{n} k_i x_i = k_L \\
x \geq 0
\]

The optimal portfolio will be a barbell portfolio (i.e., a portfolio consisting of a short maturity and a long maturity bond). The reason is that long bonds are most efficient in maximizing dollar duration times yield, and that a short bond is more efficient in reducing overall dollar duration to its target value.

The cash flows will be very disperse.
1. Ensures high positive convexity so immunized portfolio attains its minimum at current yield levels
2. Ensures that liabilities can be retired as they occur
The immunized portfolio is highly exposed to shape risk. Suppose that the liabilities consist of a single payment in ten years and the optimal asset portfolio consists of a ten-year zero coupon bond and cash. If the yield curve tilts so yields rise more than proportionally on long bonds than on short bonds, the asset portfolio will lose value more rapidly than the liability, thus causing negative net worth.

Solution is to modify the problem is to minimize the portfolio cash flow dollar convexity.

\[
Q_i = \sum_{t=1}^{T} \frac{t(t+1)C_i}{(1+r)^t}
\]

Portfolio dollar convexity

\[
Q_x = \sum_i Q_i x_i
\]

\[
\max \sum_i Q_i x_i
\]

subject to the following constraints

\[
\sum_i P_i x_i = P_L
\]

\[
\sum_i k_i x_i = k_L
\]

\[
\sum_i Q_i x_i \geq Q_L
\]

\[
x \geq 0
\]
Convexity constraint is a remedy to volatility risk. Because of bond convexity, variations in bond yields will cause the immunization conditions to break. By forcing the net portfolio convexity to be non-negative, net worth has a minimum at current yields. If an upside due to volatility is not desired, but tracking performance is required, the above convexity minimization is appropriate. Convexity constraints become the more important when securities with embedded options are available in the selection universe.

Immunization is a dynamic strategy implying that portfolio changes must be made over time. The differential effect on asset and liability dollar duration from the passage of time is often denoted as duration drift. When dollar durations drift apart, one side will be more exposed to interest-rate risk than the other. Hence, in order to maintain dollar duration equality, the portfolio must therefore be periodically rebalanced. The need to rebalance implies that an immunized portfolio will be subject to liquidity risk. The most efficient approach for the control of liquidity risk is to demand that securities be bought in large lot sizes since these are more liquid than small lot sizes.

Let $y_i$ be a binary variable for security $i$, let $l_i$ denote the minimum lot sized to purchase and let $u_i$ be the maximum permissible lot size (e.g., diversification requirements). The liquidity constraint is as follows

$$l_i y_i \leq x_i \leq u_i y_i$$

The resultant optimization problem becomes a mixed-integer programming problem (MIP).
Factor immunization is an enhanced immunization technique that addresses shape risk. This is achieved by relaxing the implicit assumption in the immunization model that the term structure of interest is flat and only shifts in parallel.

We use a model to value bonds using the spot rate yield curve (i.e., the yield curve of zero coupon bonds or, in the U.S., stripped Treasury securities).

$$P = \sum_{t \in T} \frac{C_t}{(1 + r_t)}$$

Differentiate the bond price to obtain the generalized interest rate sensitivity:

$$dP_i = -\sum_{t \in T} \left[ \frac{tC_i}{(1 + r_t)} \right] dr_t$$

One can postulate a model that relates changes in the term structure to common factors such as the one we developed using principal component analysis:

$$dr_t = \sum_{j \in J} a_{ij} dF_j = \sum_{j \in J} a_{ij} U_j$$

Factor loadings

$$L_{ij} = \frac{\partial P_i}{\partial F_j} = -\sum_{t \in T} \frac{t a_{ij} C_t}{(1 + r_t)^t}$$

An optimally factor-immunized portfolio can be established by maximizing portfolio yield subject to present value equality and factor dollar duration matching on asset and liability for all factors.
Fixed Income Portfolio Design Models

\[
\begin{align*}
\max \sum_{i \in \mathcal{L}} k_i x_i \\
\text{subject to the following constraints} \\
\sum_{i \in \mathcal{L}} P_i x_i &= P_L \\
\sum_{i \in \mathcal{L}} f_{i,j} x_i &= f_{i,j} \quad \forall j \in J \\
x &\geq 0
\end{align*}
\]

This is a linear programming problem or MIP if liquidity constraints are incorporated

---

Fixed Income Portfolio Design Models

Bond dedication is the problem of making asset cash flows look as closely as possible to the liability cash flows. The basic idea is to establish a portfolio with cash flows which always suffice to cover liability payments, if necessary, after reinvestment to the liability dates. To eliminate reinvestment risk, a conservative reinvestment rate (possibly 0%) is assumed. Should actual reinvestment rates turn out to be higher than the conservative rate, the strategy will gradually generate a surplus.

Corporations use bond dedication to take corporate bonds (the liability) and make the payments from a portfolio. This is called defeasance.

---

Fixed Income Portfolio Design Models

- \( \tau \): an index of liability payment dates from the set \( \mathcal{T} \)
- \( \Delta \tau \): length of time between liability payment dates \( \tau - 1 \) and \( \tau \in \mathcal{T} \)
- \( L_\tau \): liability payment at time \( \tau \)
- \( \rho \): Reinvestment rate
- \( D_c \): Reinvested value of bond cash flows
- \( s_\tau \): Cash holdings (surplus)
Fixed Income Portfolio Design Models

Reinvested Value of Bond Cash Flows

\[ D_t = \sum_{i=1}^{T} \frac{C_i}{(1 + \rho)^i} \]

Cash flow balance equation

\[ \sum_{i \in T} D_i x_i + s_{t-i} (1 + \rho)^{t-i} = L_t + s_t \]

Imposing non-negativity constraints on the surplus enforces the precedence of asset cash flows to liability cash flows and dedication is achieved.

Fixed Income Portfolio Design Models

\[ \min \sum_{i \in T} P_i x_i + s_0 \]

subject to the following constraints

\[ \sum_{i \in T} D_i x_i + s_{t-i} (1 + \rho)^{t-i} = L_t + s_t \quad \forall \tau \in T_i \]

\[ s_i \geq 0 \quad \forall \tau \in T_i \]

\[ x \geq 0 \]

One problem in the dedication strategy is that it assumes that liabilities are known with certainty or that good estimates are available for the full time span. For some investors, they know the liabilities for a relatively short time period and have good estimates of the overall duration of liabilities. For these investors, a hybrid between immunization and dedication may be more appropriate. In the horizon matching, or combination matching, the purpose is to provide dedication for part of a liability stream and overall dollar duration matching.
Fixed Income Portfolio Design Models

\[
\begin{align*}
\min & \sum_{i \in \mathcal{I}} P_i x_i + s_0 \\
\text{subject to the following constraints} & \\
\sum_{i \in \mathcal{T}} P_i x_i + s_0 (1 + p)^{T_0} = L_0 + s, & \forall \tau \in T \subseteq T_1 \\
\sum_{i \in \mathcal{K}} K_i x_i = P & \\
s_i \geq 0, & \forall \tau \in T \subseteq T_1 \\
x \geq 0
\end{align*}
\]

The optimization problems described previously can be solved using the Excel Solver add-in. In practice, organizations use commercial optimization software GAMS, AMPL, LINDO. Free access optimization tools are available on the NEOS Server. The document immunizationModels.doc contains the GAMS models that can be rewritten as spreadsheet models.

Dynamic Investment Strategies

Please refer to the PDF file with the same name. Concepts in the paper are widely used in the design and implementation of guaranteed investment products.
Exchange Option

Please refer to the PDF file with the same name
Early work was performed by William Margrabe
Extensions to provide an investor the payoff of the better performing of n assets and a risk free alternative