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Improved Bounds for Incidences and Complexity of Many Faces in Arrangements of Circles and of Polynomial Arcs*

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Abstract

We derive improved upper bounds for the number of incidences between m points and n circles in the plane, and for the complexity of m distinct faces in an arrangement of circles. An improved incidence bound is also obtained for graphs of polynomials of any constant maximum degree.

1 Introduction

Let P be a finite set of points in the plane and C a finite set of circles. Let $I = I(P,C)$ denote the number of incidences between the points and circles. Let $I(m,n)$ denote the maximum value of $I(P,C)$, taken over all sets P of m points and sets C of n circles, and let $I'(m,n,X)$ denote the maximum value of $I(P,C)$, taken over all sets P of m points and sets C of n circles with at most X pairs of the circles intersecting.

In this paper we derive improved upper bounds for $I(m,n)$ and $I'(m,n,X)$. The previous best upper bounds were $I(m,n) = O(m^{3/5}n^{4/5}+m+n)$ [7, 13], and $I'(m,n,X) = O(m^{3/5}X^{2/5} + m + n)$ [3]. The bounds that we obtain are:

$$I(m,n) = \begin{cases} O(m^{2/3}n^{2/3} + m) & \text{for } m \ge n^{3/2} \\ O(m^{4/7}n^{17/21} + n) & \text{for } m \le n^{3/2} \end{cases}$$

and

$$I'(m,n,X) = \begin{cases} O(m^{2/3}X^{1/3} + m) & \text{for } m \ge X^{1/2}n^{1/2} \\ O(m^{4/7}X^{8/21}n^{1/21} + n) & \text{for } m \le X^{1/2}n^{1/2}. \end{cases}$$

The bound on $I(m,n)$ is worst-case tight when $m \ge n^{3/2}$. This follows from the construction of $\Theta(m^{2/3}n^{2/3})$ incidences between m points and n lines (see, e.g., [9]) which, after applying an

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inversion to the plane, becomes a configuration with $\Theta(m^{2/3}n^{2/3})$ incidences between m points and n circles.

If C has the additional property that every two circles in C intersect, then we have the further improved bound:

$$I(m,n) = \begin{cases} O(m^{2/3}n^{2/3} + m) & \text{for } m \geq n^{5/4} \\ O(m^{6/11}n^{9/11} + n) & \text{for } m \leq n^{5/4}. \end{cases}$$

Here the bound is tight for $m \geq n^{5/4}$.

A related quantity that was studied by Clarkson *et al.* [7] is the combinatorial complexity of m distinct faces in an arrangement of n circles in the plane. The combinatorial complexity of a face is the number of arrangement edges that appear on its boundary. Let C be a finite set of circles and let P be a finite set of points, not lying on any circle of C. We denote by $K(P,C)$ the combined complexity of the faces of the arrangement $\mathcal{A}(C)$ that contain points of P. We let $K(m,n)$ denote the maximum value of $K(P,C)$, taken over all sets P of m points and sets C of n circles, and let $K'(m,n,X)$ denote the maximum value of $K(P,C)$, taken over all sets P of m points and sets C of n circles such that at most X pairs of the circles intersect.

The previous best upper bounds on $K(m,n)$ for arbitrary circles was $O(m^{3/5}n^{4/5} \cdot 2^{2\alpha(n)/5} + n)$ [7], where $\alpha(\cdot)$ is the inverse Ackermann function. If all the circles in C are congruent (the so-called *unit circles* case) then one has $K(m,n) = O(m^{2/3}n^{2/3}(\alpha(n))^{1/3} + n)$ [7].

In this paper we also derive improved bounds for $K(m,n)$ and for $K'(m,n,X)$. Unlike the case of incidences, the improvements are less dramatic, and only consist of the removal of the factors involving $\alpha(n)$, and of making the bounds depend on X. Specifically, we show that $K'(m,n,X) = O(m^{2/3}X^{1/3} + n)$ for unit circles and $O(m^{3/5}X^{2/5} + n)$ for arbitrary circles. Substituting $X = O(n^2)$, this also yields $K(m,n) = O(m^{2/3}n^{2/3} + n)$ for unit circles and $O(m^{3/5}n^{4/5} + n)$ for arbitrary circles. The best known lower bound for either quantity is again related to the construction of n lines and m points with $\Theta(m^{2/3}n^{2/3})$ incidences and implies $K(m,n) = \Omega(m^{2/3}n^{2/3} + m + n)$ for unit (and thus also arbitrary) circles, so that our upper bound for unit circles is tight.

Our results depend crucially on a theorem of Tamaki and Tokuyama [16] on cutting circles into pseudo-segments; it is reviewed and extended in Section 2.2. Recently, Chan [4] has extended this result to the case of graphs of polynomials of any constant maximum degree. Using Chan's bound and extending, in a straightforward manner, the proof technique used for the case of circles, we also obtain bounds for the number of incidences between m points and n such graphs. These bounds improve those obtained earlier in [13].

2 Preliminaries

2.1 Crossing lemmas

A (not necessarily simple) graph is said to be *drawn* in the plane if its vertices are mapped to distinct points in the plane, and each of its edges is mapped to a curve connecting the points corresponding to the end vertices of the edge. In case of multigraphs (i.e., graphs where two vertices may be connected by more than one edge), different edges between the same pair of vertices are mapped to different curves connecting the same pair of corresponding points. We further require that no curve passes through any other vertex and that each pair of curves meet a finite number of times. A *crossing* between two curves is a point at which their relative interiors intersect transversally. An *edge-crossing* in (the drawing of) the graph is a pair of crossing edges.

Lemma 2.1 (Leighton [10]; Ajtai et al. [2]; see also Pach and Agarwal [12]). *Any plane drawing of a simple graph G with e edges and n vertices must have $\Omega(e^3/n^2)$ edge-crossings, provided that $e \geq 4n$. Equivalently, if G can be drawn in the plane with X edge-crossings then $e = O(n^{2/3}X^{1/3} + n)$.*

We will also need the following extension of the lemma, due to Székely, to the case of multigraphs.

Lemma 2.2 (Székely [15]). *Any plane drawing of a multigraph G with e edges and n vertices, so that no pair of vertices is connected by more than k edges, must have $\Omega(e^3/(n^2k))$ edge-crossings, provided that $e \geq 5nk$. Equivalently, if G can be drawn in the plane with X crossings then $e = O(n^{2/3}X^{1/3}k^{1/3} + nk)$.*

Slightly abusing the notation, we will sometimes not distinguish between the vertices of a graph and the corresponding points in its plane drawing or between a (multi-)graph edge and the curve that represents it in the drawing.

2.2 Cutting circles into pseudo-segments

The basic result that is crucial to our analysis is:

Theorem 2.3 (Tamaki and Tokuyama [16, Theorem 6.1]). *Let C be a collection of n circles in the plane. Then the circles of C can be cut into a total of $O(n^{5/3})$ arcs, so that each pair of these arcs intersect at most once.*

Remark. A *lens* in $\mathcal{A}(C)$ is a pair of arcs of different circles with common endpoints (see also below). Inspecting the analysis of Tamaki and Tokuyama [16], the crucial parameter, denoted by $\nu_1(C)$, that governs the number of cuts, is the maximum cardinality of a set of lenses in $\mathcal{A}(C)$, no two of which have overlapping arcs. As can be seen from the proof of Theorem 3.1, given below, all we need is an upper bound on $\nu_1(C)$. The analysis of [16] starts with deriving the bound $\nu_1(C) = O(n^{5/3})$, and then uses several additional nontrivial steps to conclude from it the same asymptotic bound on the number of cuts. In other words, our analysis relies on only a small, and rather simple, portion of the analysis of [16] (even though this does not yield a better bound, that is, $\nu_1(C)$ and the number of cuts behave the same asymptotically).

We will also need the following extensions of this result, which has also been observed independently by Chan [4].

Lemma 2.4. *Let C be a set of n circles with at most X intersecting pairs. Then the circles of C can be cut into $O(n^{1/3}X^{2/3} + n)$ arcs so that each pair of arcs intersect at most once.*

Proof. We assume that $X \geq n$. Otherwise, cutting each circle between each consecutive pair of points of its intersection with the other circles yields a collection of $O(n)$ arcs with the desired property. Put $r = \lceil n^2/X \rceil$, and let R be a random sample of r circles from C. Let $\mathcal{A}^*(R)$ denote the vertical decomposition of the arrangement $\mathcal{A}(R)$. It is obtained from $\mathcal{A}(R)$ by drawing a vertical segment through every vertex of $\mathcal{A}(R)$ and through every leftmost and rightmost point of a circle of R and extending it upward and downward until the first intersection with an edge of $\mathcal{A}(R)$ or to infinity, otherwise. This is a decomposition of the plane into "pseudo-trapezoidal" cells, whose expected number is $O(r + (r/n)^2X) = O(r)$. For each cell τ, let C_τ denote the subset of circles of C that cross τ and put $n_\tau = |C_\tau|$. We first cut each circle of $C \setminus R$ at the points where it enters and leaves cells of $\mathcal{A}^*(R)$, and then, for each cell τ, we cut further the portions of the circles of C_τ that lie inside τ into $O(n_\tau^{5/3})$ subarcs, as in Theorem 2.3. We also cut the circles of R at vertices of $\mathcal{A}^*(R)$. The total number of arcs is thus $O(r) + \sum_\tau O(n_\tau^{5/3})$. Using the results of [6], the expected value of this sum is

$$O(r) + O(r) \cdot O((n/r)^{5/3}) = O(n^{5/3}/r^{2/3}) = O(n^{1/3}X^{2/3}),$$

as asserted. $\square$

Suppose next that C is a collection of n circles, every pair of which intersect. In this case we can improve the preceding results, as follows.

Theorem 2.5. *If C is a collection of n circles, every pair of which intersect, then one can cut the circles of C into $O(n^{3/2})$ subarcs, every two of which intersect at most once.*

Proof. Let γ, γ' be two circles in C, which intersect at two points u, v. As already defined, the union of an arc of γ and an arc of γ', each connecting u and v, is called a *lens*. Two lenses are said to *overlap* if they have a pair of overlapping arcs (along the same circle). The *interior* of the lens is the open region enclosed by its two circle arcs. We say that another circle δ *crosses* the lens ℓ if it meets both the lens itself and its interior. A lens is *empty* if no other circle crosses it. We use the following result of [3]:

Lemma 2.6. *The number of empty lenses in an arrangement of n pairwise-intersecting circles is $O(n)$.*

Lemma 2.7. *If C is a collection of n pairwise-intersecting circles then $\nu_1(C) = O(n^{3/2})$.*

Proof. Define the *level* of a lens to be the number of circles of C that cross it. Empty lenses are precisely the 0-level lenses. By a straightforward application of the Clarkson-Shor probabilistic analysis technique [6], the number L_k of lenses at level at most k is $O(k^2)$ times the number of empty lenses in a random sample of n/k circles of C. Using Lemma 2.6, $L_k = O(k^2 \cdot (n/k)) = O(nk)$.

Let $\mathcal{L}^*$ be any collection of pairwise nonoverlapping lenses with levels greater than k. Let $\ell \in \mathcal{L}^*$, and let γ, γ' be the two circles whose arcs form ℓ. Then ℓ can be naturally associated with at least k ordered pairs of crossing circles of the form (γ, δ) or (γ', δ), where δ is a circle that crosses ℓ. We claim that any such pair (γ, δ) can be associated with at most four lenses of $\mathcal{L}^*$. Indeed, let u, v denote the points of intersection of γ and δ. There can be at most two lenses in $\mathcal{L}^*$ that have an arc along γ and contain u, and at most two lenses in $\mathcal{L}^*$ that have an arc along γ and contain v. No other lens in $\mathcal{L}^*$ can be associated with (γ, δ).

Hence the maximum number of pairwise nonoverlapping lenses of level greater than k is $O(n^2/k)$. This, plus the bound $O(nk)$ on all lenses of level at most k, yields the bound $\nu_1(C) = O(nk + n^2/k)$. Substituting $k = n^{1/2}$ yields the asserted bound. $\square$

Inspecting further the analysis of [16], one can show that Lemma 2.7 implies that the total number of cuts required is also $O(n^{3/2})$. $\square$

Remark. As mentioned above, Lemma 2.7 is all we need for our argument. Nevertheless, Theorem 2.5 is of independent interest, which is why we derived it as well.

2.3 The case of polynomial curves

Let Γ be a collection of n curves that are graphs of polynomial functions of constant maximum degree s. In a recent paper, Chan [4] studied the problem of cutting the curves of Γ into arcs, each pair of which intersects at most once. The following bound, which is lower by a polylogarithmic factor than the bound given in [4], can be derived in a straightforward manner by the technique of [4]; it is smaller because we do not require the additional property that the resulting arcs be *extendible* to a collection of pseudolines, a property needed in Chan's application.

Theorem 2.8 (Chan [4]). *Any collection of n curves that are graphs of polynomial functions of constant maximum degree s can be cut into $O(n^{2-1/3^{s-1}})$ arcs, each pair of which intersect at most once.*

3 Improved Bounds for Incidences Between Points and Circles

3.1 Improved bounds for many points

Let P be a set of m points in the plane and C a set of n circles with X intersecting pairs. Put $I = I(P, C)$.

By Lemma 2.4, we can cut the circles of C into $O(n^{1/3}X^{2/3})$ arcs, so that each pair of arcs intersect at most once. Let C' denote the resulting collection of arcs.

We draw a graph G in the plane whose vertices are the points of P, and whose edges connect pairs of points u, v that are consecutive along an arc of C'. We assume that each arc of C' contains at least two points of P; the contribution of the remaining arcs to I is at most $|C'|$. We also assume that any circle that has not been cut at all contains at least three points of P; the contribution of the other circles to I is at most $2n$. It is easily seen that the number e of edges of G is at least $I - cn^{1/3}X^{2/3} - 2n$, for some constant c. By construction, the graph G is simple, so the Crossing Lemma 2.1 implies that $I - cn^{1/3}X^{2/3} - 2n = O(m^{2/3}X^{1/3} + m)$. That is, $I = O(m^{2/3}X^{1/3} + n^{1/3}X^{2/3} + m + n)$. In other words, we have shown:

Theorem 3.1. *The maximum number of incidences between m points and n circles in the plane, with X crossing pairs of circles, is*

$$I'(m, n, X) = O(m^{2/3}X^{1/3} + n^{1/3}X^{2/3} + m + n). \tag{1}$$

In particular, the maximum number of incidences between m points and n circles in the plane is

$$I(m, n) = O(m^{2/3}n^{2/3} + n^{5/3} + m). \tag{2}$$

Remark. The bound in (2) is tight when $m \geq n^{3/2}$. To see this, suppose that $m = O(n^2)$ (otherwise the bound is trivially tight), and construct a collection of m points and n lines with $\Theta(m^{2/3}n^{2/3})$ incidences between them (see, e.g., [9]). By applying an appropriate inversion transformation to the plane, this configuration is mapped to a collection of m points and n circles (which, by the way, all pass through a common point) with $\Theta(m^{2/3}n^{2/3})$ incidences between them, showing that the above bound is indeed tight.

Remark. Theorem 3.1 raises several interesting open problems:

(i) Can the bound of [16] on the number of cuts be improved? The best known lower bound on this quantity is only $\Omega(n^{4/3})$ [16], which also implies that the reasoning in the proof of Theorem 3.1 cannot produce a bound better than $I(m,n) = O(m^{2/3}n^{2/3} + n^{4/3} + m)$.

(ii) Another result of [3] shows that the number of empty lenses in an arrangement of n unit circles is $O(n^{4/3}\log n)$. Following the reasoning of the proof of Lemma 2.7, we conclude that a collection of n unit circles can be cut into $O(n^{8/5}\log^{3/5} n)$ arcs, each pair of which intersect at most once. This improvement is interesting but is not relevant to our theme, since for unit circles the number of incidences is known to be $O(m^{2/3}n^{2/3} + m + n)$. At any rate, we conjecture that both this bound and the bound in Lemma 2.7 are far from being tight.

(iii) The cutting of [16] is too 'aggressive', in the sense that it performs too many cuts in order to guarantee that every pair of surviving arcs intersect at most once. How many cuts are needed to guarantee only that the maximum cardinality of a pencil (a collection of surviving arcs that pass through the same pair of points) is at most k, for a prescribed parameter k? In Appendix C we show that $O(n^{5/3}/k^{4/3})$ cuts suffice to ensure this property. We remark that for our application a more appropriate quantity to bound is the number of cuts needed to ensure that the number of surviving arcs that connect the same pair of points in P so that the points are *consecutive* along these arcs, is at most k. We still do not know how to exploit this additional constraint to improve the above-mentioned bound.

3.2 Improved bounds for any number of points

The bounds obtained above are $O(m^{2/3}n^{2/3} + m)$ when $m \geq n^{3/2}$ and $O(m^{2/3}X^{1/3} + m)$ when $m \geq n^{1/2}X^{1/2}$, but are larger for smaller values of m. Our next step is to use a partitioning of dual space to improve the bound for smaller values of m.

We use the following transformation: A circle γ in the plane, of radius ρ and centered at (a,b), is mapped to the point $\gamma^*(a, b, \rho^2 - a^2 - b^2) \in \mathbb{R}^3$, and a point $p(\xi, \eta)$ in the plane is mapped to the plane $p^* : \; z = -2\xi x - 2\eta y + (\xi^2 + \eta^2)$ in $\mathbb{R}^3$. As is easily verified, a point p lies on a circle γ if and only if the dual plane p^* contains the dual point γ^*. Let P^* denote the set of planes dual to the points of P and let C^* denote the set of points dual to the circles of C. No three planes of P^* pass through a common line, as all planes of P^* are tangent to the paraboloid $z = -x^2 - y^2$.

We need the following well-known result of Matoušek:

Theorem 3.2 (Matoušek [11]). *Let A be a set of n points in $\mathbb{R}^d$ and $1 \leq r \leq n$ a given parameter. Then A can be partitioned into $q \leq 2r$ subsets, $A_1, \ldots, A_q$, so that, for each i, $|A_i| \leq n/r$, and A_i is contained in a (possibly lower-dimensional) simplex Δ_i, so that no hyperplane* crosses *(i.e., intersects but does not contain) more than $O(r^{1-1/d})$ of these simplices.*

Apply Theorem 3.2 to C^* (with $d = 3$), with a value of r that will be fixed shortly, to obtain a partitioning of C^* into subsets $C_1^*, \ldots, C_q^*$, where $q \leq 2r$, with the above properties. Let C_i be the subset of circles in C that are dual to the points of C_i^*, let P_i denote the set of points of P whose dual planes cross the corresponding simplex Δ_i, and put $m_i = |P_i|$, for $i = 1, \ldots, q$. We have $\sum_{i=1}^{q} m_i = O(mr^{2/3})$.

Let $p \in P$ be a point that is incident to at least one circle in C_i, for some i. Then either the dual plane p^* crosses Δ_i, that is, $p \in P_i$, or p^* contains Δ_i. The latter case can arise only when Δ_i (and C_i^*) has lower dimension. If Δ_i has dimension 2 then there can be at most one point $p \in P$ whose dual plane contains Δ_i. If Δ_i is one-dimensional then, as noted above, there can be at most 2 points in P whose dual planes contain Δ_i. We may rule out the case that Δ_i is zero-dimensional, because then C_i is a singleton, and the construction of [11] never produces any singleton subset. Hence, the total number of incidences between P and C that fall into these degenerate categories is at most $2n$. In other words, we have shown that

$$I(P, C) \leq 2n + \sum_{i=1}^{q} I(P_i, C_i).$$

Applying Theorem 3.1 to each $I(P_i, C_i)$, we thus obtain

$$I(P, C) = O\left(n + \sum_{i=1}^{q}\left(m_i^{2/3}(n/r)^{2/3} + (n/r)^{5/3} + m_i\right)\right),$$

which, using Hölder's inequality, becomes

$$I(P, C) = O\left(n + \left(\sum_{i=1}^{q} m_i\right)^{2/3} \cdot r^{1/3} \cdot (n/r)^{2/3} + n^{5/3}/r^{2/3} + \sum_{i=1}^{q} m_i\right) =$$

$$O\left(n + \left(mr^{2/3}\right)^{2/3} \cdot r^{1/3} \cdot (n/r)^{2/3} + n^{5/3}/r^{2/3} + mr^{2/3}\right) =$$

$$O\left(m^{2/3}n^{2/3}r^{1/9} + n^{5/3}/r^{2/3} + mr^{2/3} + n\right).$$

Choose $r = n^{9/7}/m^{6/7}$. We note that $r \geq 1$ when $m \leq n^{3/2}$, which is the range under consideration, and that $r \leq n$ provided that $m \geq n^{1/3}$, which we may also assume, since for smaller values of m we have $I(P, C) = O(n)$, as follows, e.g., from [7]. Substituting the value of r yields $I(P, C) = O(m^{4/7}n^{17/21} + n)$. We thus have:

Theorem 3.3. *The maximum number of incidences between m points and n circles in the plane is*

$$I(m, n) = \begin{cases} O(m^{4/7}n^{17/21} + n) & m \leq n^{3/2} \\ O(m^{2/3}n^{2/3} + m) & m \geq n^{3/2}. \end{cases} \tag{3}$$

Remark. Our analysis only requires that the bound $\sum_i m_i = O(mr^{2/3})$ holds, rather than the stronger property, provided in Theorem 3.2, that each m_i is $O(r^{2/3})$. This weaker property can be derived in a manner that is somewhat simpler than the analysis of [11].

We next extend the above analysis to derive an improved bound on $I'(m, n, X)$. We argue as in the proof of Lemma 2.4. That is, we fix $r = \lceil n^2/X \rceil$. We may assume, as above, that $X = \Omega(n)$, for otherwise, trivially, $I(m, n) = O(m + n)$. Let R be a random sample of r circles from C. Let $\mathcal{A}^*(R)$ denote the vertical decomposition of the arrangement $\mathcal{A}(R)$, whose expected size is, as above, $O(r)$. For each (open) cell τ, let C_τ denote the subset of circles of C that either cross τ or contribute an arc to $\partial\tau$, and let P_τ denote the set of points of P in the closure of τ. Put $m_\tau = |P_\tau|$ and $n_\tau = |C_\tau|$. By construction, $I(P, C) \leq \sum_\tau I(P_\tau, C_\tau)$. Since a point can belong to at most two cells of which it is not a vertex, it follows that the expected value of $\sum_\tau m_\tau$ is at most $2m + O(r) = O(m)$, provided that $r \le m$. Hence $I(P, C)$ is at most proportional to the expected value of

$$\begin{aligned}
&\sum_{m_\tau \ge n_\tau^{3/2}} \left(m_\tau^{2/3} n_\tau^{2/3} + m_\tau\right) + \sum_{m_\tau < n_\tau^{3/2}} \left(m_\tau^{4/7} n_\tau^{17/21} + n_\tau\right) \\
&\qquad = O(m) + \sum_\tau n_\tau + \sum_{m_\tau \ge n_\tau^{3/2}} m_\tau^{2/3} n_\tau^{2/3} + \sum_{m_\tau < n_\tau^{3/2}} m_\tau^{4/7} n_\tau^{17/21} \\
&\qquad \le O(m) + \sum_\tau n_\tau + \sum_\tau m_\tau^{2/3} n_\tau^{2/3} + \sum_\tau m_\tau^{4/7} n_\tau^{17/21} \\
&\qquad \le O(m) + \sum_\tau n_\tau + \left(\sum_\tau m_\tau\right)^{2/3} \left(\sum_\tau n_\tau^2\right)^{1/3} + \left(\sum_\tau m_\tau\right)^{4/7} \left(\sum_\tau n_\tau^{17/9}\right)^{3/7} \\
&\qquad \le O(m) + O\left(r \cdot \frac{n}{r}\right) + O(m^{2/3}) \cdot O\left(r \cdot \left(\frac{n}{r}\right)^2\right)^{1/3} + O(m^{4/7}) \cdot O\left(r \cdot \left(\frac{n}{r}\right)^{17/9}\right)^{3/7} \\
&\qquad = O\left(\frac{m^{2/3} n^{2/3}}{r^{1/3}} + \frac{m^{4/7} n^{17/21}}{r^{8/21}} + m + n\right),
\end{aligned}$$

where we have used Hölder's inequality and an estimate due to Clarkson and Shor [6] on the expected value of sums of the form $\sum_\tau n_\tau^\alpha$. Substituting the value of r, and assuming it to be at most m, we conclude that

$$I'(m, n, X) = O\left(m^{2/3} X^{1/3} + m^{4/7} X^{8/21} n^{1/21} + m + n\right).$$

If $r > m$ then the expected value of $\sum_\tau m_\tau$ is $O(r)$. If we substitute this bound in the above chain of inequalities, we obtain that

$$I(P, C) = O(r + n + r^{1/3} n^{2/3} + r^{4/21} n^{17/21}) = O(n),$$

since $r \le n$. Hence the above bound for $I'(m, n, X)$ applies in all cases.

We can summarize the preceding arguments as follows.

Theorem 3.4.

$$I'(m, n, X) = \begin{cases} O(m^{2/3} X^{1/3} + m) & m \ge X^{1/2} n^{1/2} \\ O(m^{4/7} X^{8/21} n^{1/21} + n) & m \le X^{1/2} n^{1/2}, \end{cases}$$

assuming that $X \ge n$*; otherwise* $I'(m, n, X) = O(m + X) = O(m + n)$.

3.3 Improved bound for the case of pairwise intersecting circles

Suppose that C is a collection of n circles, every pair of which intersect. In this case we can obtain a further improved bound on $I(P, C)$, by applying the analysis presented in the preceding subsections, replacing the bound $O(n^{5/3})$ on the number of cuts by the improved bound $O(n^{3/2})$

provided in Theorem 2.5. We then have, using the same notation as in the proof of Theorem 3.3,

$$\begin{aligned}
I(P,C) &\le 2n + \sum_{i=1}^{q} I(P_i, C_i) \\
&= O\left(n + \sum_{i=1}^{q}\left(m_i^{2/3}(n/r)^{2/3} + (n/r)^{3/2} + m_i\right)\right) \\
&= O\left((mr^{2/3})^{2/3} r^{1/3}(n/r)^{2/3} + n^{3/2}/r^{1/2} + mr^{2/3} + n\right) \\
&= O\left(m^{2/3}n^{2/3}r^{1/9} + n^{3/2}/r^{1/2} + mr^{2/3} + n\right).
\end{aligned}$$

We now choose $r = n^{15/11}/m^{12/11}$. We note that $r \ge 1$ when $m \le n^{5/4}$, which is now the range under consideration (this is the range where $m^{2/3}n^{2/3} \le n^{3/2}$), and that $r \le n$ provided that $m \ge n^{1/3}$, which we may also assume, as above. Substituting the value of r, we obtain $I(P,C) = O(m^{6/11}n^{9/11} + n)$. We have thus shown:

Theorem 3.5. *The maximum number of incidences between* m *points and* n *pairwise-intersecting circles in the plane is*

$$I(m,n) = \begin{cases} O(m^{2/3}n^{2/3} + m) & m \ge n^{5/4} \\ O(m^{6/11}n^{9/11} + n) & m \le n^{5/4}. \end{cases} \tag{4}$$

3.4 Incidences between points and graphs of polynomials

Let P be a set of m points and Γ be a collection of n curves that are the graphs of polynomials of degree at most s, for some fixed parameter $s \ge 1$. We wish to bound the number of incidences $I(P,\Gamma)$ between the points of P and the curves in Γ. We set $I(m,n) = \max\ I(P,\Gamma)$, where the maximum is taken over all sets P, Γ as above.

Our first step is similar to the analysis in Theorem 3.1. That is, we apply Chan's result, given in Theorem 2.8, to obtain a cutting of the curves in Γ into $O(n^{2-1/3^{s-1}})$ arcs, each pair of which intersect at most once. Continuing as in the proof of Theorem 3.1, we readily obtain the first bound

$$I(m,n) = O(m^{2/3}n^{2/3} + m + n^{2-1/3^{s-1}}). \tag{5}$$

As above, this bound is tight for $m \ge n^{2-1/(2\cdot 3^{s-2})}$.

To obtain an improved bound for smaller values of m, we apply the following duality transform. Each curve γ of the form $y = a_0 + a_1x + a_2x^2 + \cdots + a_sx^s$ is mapped to the point $\gamma^*(a_0, a_1, \ldots, a_s) \in \mathbb{R}^{s+1}$. Each point $p(\xi,\eta)$ is mapped to the hyperplane $p^*:\ x_0 + \xi x_1 + \xi^2 x_2 + \cdots + \xi^s x_s = \eta$. Clearly, incidences between points and curves are mapped to incidences between the corresponding dual hyperplanes and points.

Now apply Theorem 3.2 to the set Γ^* of points dual to the curves in Γ, with a parameter r that will be chosen shortly. We obtain a partition of Γ^* into $q \le 2r$ subsets, $\Gamma_1^*, \ldots, \Gamma_q^*$, each containing at most n/r points, so that no hyperplane can be incident to points in more than $O(r^{s/(s+1)})$ subsets, except for subsets that the hyperplane fully contains. It is easily checked that if a subset has at least two points (a condition that always holds in the construction provided in Theorem 3.2) then the number of hyperplanes of the above form that fully contain the set is at most s. (In the primal plane, this statement asserts that two distinct polynomials of degrees at most s cannot coincide at more than s points.) It follows that the number of incidences of the latter kind is at most $sn = O(n)$, so we may disregard them in what follows.

Applying the bound of (5) to each of the subsets Γ_i that correspond to the dual sets Γ_i^*, and summing over all such subsets, we obtain the bound

$$I(m,n) = O\left(n + \sum_{i=1}^{q} \left(m_i^{2/3}(n/r)^{2/3} + m_i + (n/r)^{\beta_s}\right)\right),$$

where $\beta_s = 2 - 1/3^{s-1}$ and where m_i is the number of hyperplanes dual to the points of P that cross the simplex containing Γ_i. Using Hölder's inequality and the fact that $\sum_i m_i = O(mr^{s/(s+1)})$, we have

$$\begin{aligned} I(m,n) &= O\left((mr^{s/(s+1)})^{2/3} r^{1/3} (n/r)^{2/3} + mr^{s/(s+1)} + \frac{n^{\beta_s}}{r^{\beta_s - 1}}\right) \\ &= O\left(m^{2/3} n^{2/3} r^{(s-1)/(3(s+1))} + mr^{s/(s+1)} + \frac{n^{\beta_s}}{r^{\beta_s - 1}}\right). \end{aligned}$$

We now put

$$r = \frac{n^{\frac{4-\frac{1}{3^{s-2}}}{4-\frac{1}{3^{s-2}}-\frac{2}{s+1}}}}{m^{\frac{2}{4-\frac{1}{3^{s-2}}-\frac{2}{s+1}}}}.$$

We may assume that $n^{1/(s+1)} \le m \le n^{2-1/(2\cdot 3^{s-2})}$. Indeed, for $m > n^{2-1/(2\cdot 3^{s-2})}$ we already have a tight bound, and if $m < n^{1/(s+1)}$ then the number of polynomials that pass through at least $s+1$ of the given points is at most $O(m^{s+1}) = O(n)$, which is also easily seen to bound the number of incidences between these polynomials and the given points, whereas any other polynomial has at most s incidences with the given points, for an overall bound of $O(n)$ incidences. It is easily verified that in the assumed range for m we have $1 \le r \le n$. Substituting this value of r, the above bound becomes

$$I(m,n) = O\left(m^{\frac{2-\frac{2}{3^{s-1}}}{4-\frac{1}{3^{s-2}}-\frac{2}{s+1}}} n^{1-\frac{\frac{2}{s+1}\left(1-\frac{1}{3^{s-1}}\right)}{4-\frac{1}{3^{s-2}}-\frac{2}{s+1}}} + n\right).$$

One can also verify that this bound is (slightly) better than the bound

$$I(m,n) = O(m^{(s+1)/(2s+1)} n^{2s/(2s+1)} + m + n)$$

obtained in [13] (for somewhat more general families of curves), provided that $m > n^{1/(s+1)}$, which, as above, can be assumed.

We summarize this section in the following theorem.

Theorem 3.6. *The maximum number of incidences between m points and n graphs of polynomials of constant maximum degree s is*

$$I(m,n) = \begin{cases} O\left(m^{\frac{2-\frac{2}{3^{s-1}}}{4-\frac{1}{3^{s-2}}-\frac{2}{s+1}}} n^{1-\frac{\frac{2}{s+1}\left(1-\frac{1}{3^{s-1}}\right)}{4-\frac{1}{3^{s-2}}-\frac{2}{s+1}}} + n\right) & m \le n^{2-1/(2\cdot 3^{s-2})} \\ O(m^{2/3}n^{2/3} + m) & m \ge n^{2-1/(2\cdot 3^{s-2})}. \end{cases} \tag{6}$$

4 Complexity of Many Faces

In this section we bound the maximum complexity $K(m,n)$ of m faces in an arrangement of n circles or of unit circles. Our analysis is based on Székely's technique [15], and extends an earlier application of this technique, reported, e.g., by Dey and Pach [8], for bounding the complexity of many faces in an arrangement of lines.

4.1 Complexity of many faces in arrangements of unit circles

In this subsection we prove the following theorem.

Theorem 4.1. *The combinatorial complexity of m distinct faces in an arrangement of n unit circles is $O(m^{2/3}n^{2/3}+n)$, or $O(m^{2/3}X^{1/3}+n)$, where X is the number of intersecting pairs of circles.*

Proof. Let C be a collection of n circles and P a collection of m points marking (distinct) faces in their arrangement. We aim to bound the total complexity $K=K(P,C)$ of the marked faces. Note that $m=O(X+n)$, as the total number of faces in the arrangement is at most $2X+n+1$, which follows from the observation that every unbounded face has one of the at most $2X$ arrangement vertices or one of the n rightmost points of a circle as its rightmost point. In the remainder of the proof we assume, without loss of generality, that the union of the curves of C is connected, so $X=\Omega(n)$ and $m=O(X)$; extending the analysis to cases where the union is disconnected is easy.

The analysis begins in a manner similar to that for the case of a line arrangement, as presented in [8]. We fix a point q_e in the interior of every edge e of a marked face and distinguish faces touching a circle "from the inside" and "from the outside." We construct two separate graphs G_- and G_+ to encode the two types of occurrences of a circle along a face boundary.

More precisely, if $\gamma\in C$ encloses two distinct faces f_1,f_2 in its interior, and appears along their boundaries in two respective edges, e_1,e_2, of $\mathcal{A}(C)$, so that no other marked face enclosed by γ touches γ between them (i.e., one of the two arcs of γ delimited by q_{e_1} and q_{e_2} contains no other edge of a marked face enclosed by γ; call this arc $[q_{e_1},q_{e_2}]$), we connect the corresponding marking points p_1,p_2 by an edge in G_-, drawn as follows: p_1 is connected by a Jordan arc (see below) to q_{e_1}, then q_{e_1} is connected to q_{e_2} by the arc $[q_{e_1},q_{e_2}]$ of γ, and finally q_{e_2} is connected by another Jordan arc to p_2. The Jordan arcs connecting the marking point p of a face f to its edges are chosen so that they lie completely in the interior of f, except for their endpoints, and do not cross one another. G_+ is constructed analogously, encoding the edges where faces touch a circle on the outside. See Figure 1 for an illustration.

A *face-circle incidence* is a pair (f,γ) where f is a marked face and γ is a circle appearing along ∂f. Let I be the total number of such incidences—note that it is in general strictly smaller than K, as it does not count multiple appearances of the same circle along a face boundary. We argue that in fact $K\le 2I$. To see this we recall the fact (see, e.g., [14, Theorem 5.7]) that the complexity of a single face in an arrangement of circles is at most $2\ell-2$ if ℓ circles appear along its boundary. To summarize, $I\le K\le 2I$, so we will somewhat freely switch between bounding I and bounding K.

Let γ be a circle in C. We denote by $\sigma_-=(s_1^-,s_2^-,\ldots,s_\ell^-)$ the circular sequence of marked faces that lie in the interior of γ and such that γ appears on their boundary, in the order that these boundaries appear along γ, say in clockwise direction, with the additional provision that

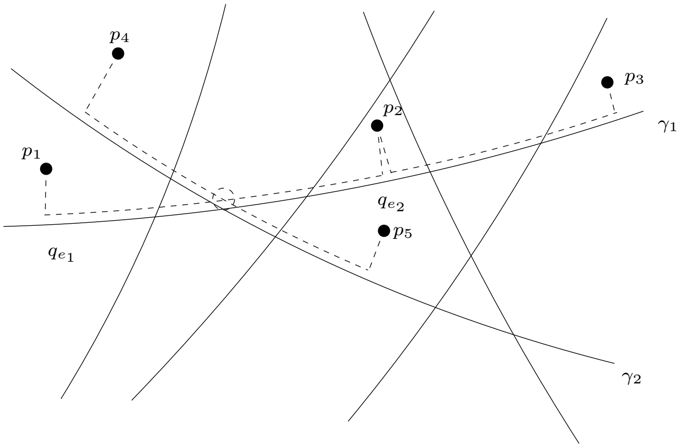

Figure 1: Three edges of the graph G_-. The edge (p_1, p_2) connecting p_1 and p_2 along the inner side of the circle γ_1 and the edge (p_4, p_5) connecting p_4 to p_5 along the inner side of γ_2 cross at an intersection point of γ_1 and γ_2.

a maximal 'run' of repeated appearances of the boundary of the same face along γ with no intervening appearances of other marked faces (enclosed by γ) is compressed in σ_- to a single symbol. We assume that γ appears on the boundary of at least three distinct marked faces enclosed by it (the remaining circles contribute at most $2n$ to the overall "inner" face-circle incidence count, and can thus be ignored). We denote by σ_+ the analogous sequence of faces that lie in the exterior of γ, provided that at least three distinct marked faces appear there.

The combined length of the sequences σ_-, σ_+ over all circles is exactly $|G_-| + |G_+| \leq K$, since, by construction, each edge of one of these graphs represents the "connection" between the occurrences of two different consecutive faces along γ and thus is also equivalently encoded by a pair of consecutive elements in the corresponding sequence σ_-, σ_+. On the other hand, the *number* of distinct symbols appearing in each sequence, summed over all sequences, is exactly $I \geq K/2$, so it is sufficient to obtain an upper bound for $|G_-| + |G_+|$.

The analysis of Clarkson *et al.* [7] implies that the multiplicity of any edge of G_- is at most two. Actually, a stronger property holds: It is impossible for two distinct faces to touch three distinct unit circles on their interior sides. Hence, Lemma 2.1 implies that $|G_-|$ is $O(m^{2/3}X_-^{1/3} + m)$, where X_- is the number of edge-crossings in G_-. Since by construction an edge-crossing in G_- is also a crossing of a pair of circles in C and no two edge-crossings can use the same circle-crossing, it follows that $X_- \leq X$ and $|G_-| = O(m^{2/3}X^{1/3} + m)$.

Handling the graph G_+ is somewhat more involved. It is shown in [7] that G_+ can be manipulated as follows. We first disregard the face of the arrangement that lies outside all circles of C, if it is marked, because it can contribute at most n to I and at most $2n - 2$ to K. Each remaining marked face is enclosed by at least one circle of C and thus has diameter at most 2. We overlay the arrangement of the circles of C with the unit grid. Each circle meets the gridlines at most 8 times, so the total number of circle arcs of the form $[q_{e_1}, q_{e_2}]$ that are part of the drawing of G_+ met by the gridlines is at most $8n$—we remove the edges corresponding to these arcs from G_+. It can now be shown (adapting the analysis given in [7]) that in what remains of G_+ the edge multiplicities are all bounded by a constant, so we can apply an analysis similar to that above to conclude that $|G_+|$ and thus also the overall face-circle incidence count is $O(m^{2/3}X^{1/3} + m + n) = O(m^{2/3}X^{1/3} + n)$ (the latter estimate follows from the fact that $m = O(X)$). This completes the proof of Theorem 4.1. $\square$

4.2 Complexity of many faces in arrangements of arbitrary circles

In this subsection we prove the following theorem.

Theorem 4.2. *The combinatorial complexity of m distinct faces in an arrangement of n arbitrary circles is $O(m^{3/5}n^{4/5} + n)$, or $O(m^{3/5}X^{2/5} + n)$, where X is the number of crossing pairs of circles.*

Proof. Let C be the given set of n circles and P a set of m points, marking the given faces of $\mathcal{A}(C)$, as above. As in the preceding subsection, it suffices to bound the number I of pairs (f, γ), where f is one of the marked faces and γ is a circle of C that appears along ∂f.

It suffices to prove the latter, X-dependent bound; the former bound follows by substituting $X = O(n^2)$. Fix a threshold parameter $k = c(X/m)^{1/5}$, where c is an absolute constant, to be chosen below. Construct the graphs G_-, G_+, defined in complete analogy to the case of unit circles, discarding from consideration in G_- (resp. in G_+) each circle of C that touches and encloses (resp. does not enclose) two or fewer distinct marked faces—each such circle contributes at most four incidences to I. Again, I is proportional to the number of edges of the graphs G_-, G_+, which are thus the quantities we aim to bound.

We focus on the graph G_-; the treatment of G_+ is fully symmetric. For the sake of clarity, we refer to the edges of G_- as *arcs*, and to any pair $\{u, v\}$ of marking points connected by at least one arc in G_- as a *simple edge*. Let the *multiplicity* of a simple edge $\{u, v\}$ be the number of arcs of G_- along which u and v are connected. We distinguish between *light edges*, whose multiplicity is at most k, and *heavy edges*, whose multiplicity is greater than k. We call an arc *light* (resp. *heavy*) if its corresponding simple edge is light (resp. heavy). Denote by L the subgraph of G_- consisting of all light arcs, and by H the subgraph of all heavy arcs.

Claim A. *The number $|L|$ of arcs in L is at most $c_1 m^{3/5}X^{2/5}$, for some absolute constant c_1.*

Proof. Lemma 2.2 yields $|L| = O(mk + m^{2/3}X^{1/3}k^{1/3})$. Substituting the value of k and recalling that $m = O(X)$, we obtain

$$|L| = O(m^{4/5}X^{1/5} + m^{3/5}X^{2/5}) = O(m^{3/5}X^{2/5}). \tag{7}$$

$\square$

For any circle $\gamma \in C$, consider all of its arcs of the form $[q_{e_1}, q_{e_2}]$ which are part of the drawing of G_-, in the circular order in which they occur along γ. If $[q_{e_u}, q_{e_v}]$ is a part of the arc $\sigma_1 = (u \to q_{e_u} \to q_{e_v} \to v)$ of G_- and its immediate neighbor in the above ordering is $[q_{e_v}, q_{e_w}]$, which is a part of an arc $\sigma_2 = (v \to q_{e_v} \to q_{e_w} \to w)$ of G_-, we say that that w is a *neighbor vertex* and σ_2 is a *neighbor arc* of $\{u, v\}$ along γ, and that σ_2 is an *extension* of σ_1 past v. Each arc of G_- has two neighbor vertices and two corresponding neighbor arcs.

We say that a heavy simple edge $\{u, v\}$ is *dense* if for at least half of the circles γ along which u and v are connected in G_-, both neighbor arcs of $\{u, v\}$ along γ are heavy. Otherwise we call $\{u, v\}$ *sparse*. As above, the notions of sparse and dense carry over to the corresponding arcs as well. Let S, D denote the two complementary subgraphs of H that consist, respectively, of all the sparse and dense arcs of H.

Claim B. *The number of arcs in S is at most $4|L| \le 4c_1 m^{3/5} X^{2/5}$.*

Proof. Let $\{u, v\}$ be a sparse simple edge with multiplicity $\ell > k$. By definition, there exist $\ell' \ge \ell/2$ arcs connecting u and v, for which one of the neighbor arcs of G_- along the same circle is light. We charge the ℓ arcs connecting u and v to these ℓ' arcs of L, charging each at most 2 units. Clearly, no arc of L is charged more than four units (it can be charged at most two units by each of its neighboring arcs). Hence, by Claim A, the total number of sparse arcs in H is at most $4|L| \le 4c_1 m^{3/5} X^{2/5}$, as asserted. □

It thus suffices to estimate the number of arcs in D. Let P_D denote the set of points in P that are incident to at least one dense simple edge, and put $m_D = |P_D|$.

Note that Claims A and B also imply that the overall number of incidences between the faces marked by the points of $P \setminus P_D$ and the circles of C is at most $5c_1 m^{3/5} X^{2/5}$. Hence, it suffices to bound the number of incidences involving the faces marked by the points of P_D.

We claim that, with an appropriate choice of the constant c in the definition of k, $m_D \le m/2$. This is proved as follows.

Claim C. *Each point $u \in P_D$ has at least $k/4$ distinct neighbors in H.*

Proof. Let $\{u, v\}$ be a simple edge of D. Then u and v are connected in G_- by $\ell > k$ arcs. Since $\{u, v\}$ is dense, at least $\ell/2 > k/2$ of these arcs have the property that their extensions past u are heavy. The other endpoints of these extensions are neighbors of u in H. They may not be all distinct, but, as follows from [7, Lemma 4.5], no neighbor can be obtained in this manner more than twice, or else it would imply the existence of three faces which simultaneously touch three circles, all on the inner side, a configuration that the analysis in [7] shows to be impossible. (An analogous statement, needed for the analysis of G_+, holds for faces touching circles on the outside.) This implies the claim. □

Now suppose to the contrary that $m_D > m/2$. Then, by Claim C, the number of simple heavy edges is at least $(m/2)(k/4)/2 = mk/16$.

Consider the subgraph H' of H in which for each simple edge we retain exactly k of its arcs. Then the number $|H'|$ of arcs in H' is at least $mk^2/16$, and its edge multiplicity is exactly k. By Lemma 2.2, the number of edge-crossings in H' is at least $c_2(mk^2/16)^3/(m^2k) = (c_2/4096)mk^5 = c_2c^5X/4096$, for some absolute constant c_2, which we make sure is impossible by choosing c large enough to force $c_2c^5/4096 > 1$. This contradiction implies that $m_D \le m/2$.

We bound the number of incidences between the faces marked by P_D and the circles of C recursively. Let $I_-(m, n, X)$ denote the maximum number of incidences between faces marked by m points and between the inner sides of n circles with at most X crossing pairs. Then we obtain the recurrence

$$I_-(m, n, X) \le I_-(m/2, n_1, X) + 5c_1 m^{3/5} X^{2/5} + O(n - n_1), \tag{8}$$

where n_1 is the number of circles enclosing and touching the boundaries of at least three distinct marked faces. The solution of (8) is easily seen to be

$$I_-(m, n, X) = O(m^{3/5} X^{2/5} + n).$$

The treatment of G_+ is fully analogous, which thus completes the proof of the theorem. □

5 Conclusion

The main observation in this paper is that two recent analysis techniques, of Székely [15] and of Tamaki and Tokuyama [16], can be combined in a straightforward manner to yield improved incidence bounds for points and circles and for other families of curves. We suspect (and hope) that similar ideas can be applied to other related problems, in two or in higher dimensions.

This paper raises many open problems. We mention here some of the more obvious ones:

- Can the Tamaki-Tokuyama bound be improved? See the remarks at the end of subsection 3.1 for this and several other related problems.
- Can the technique of this paper be adapted to tackle the problem of the number of *distinct distances* in a set of n points in the plane? The setup in this problem involves a collection of many circles that have relatively few centers (see [15] for details).
- Can one improve the upper bound, given in Theorem 4.2, on the complexity of many faces in an arrangement of arbitrary circles?
- Can one obtain improved upper bounds for the complexity of many faces in an arrangement of graphs of polynomials of fixed maximum degree?
- Find applications of the new incidence bounds obtained in this paper. Two problems to which the new bounds might be applicable are the unit distance problem in three dimensions [7] and the problem of bounding the maximum number of simplices spanned by a set of n points in $\mathbb{R}^d$ and congruent to a given simplex (see [1] for work in progress on this problem).

References

[1] P. Agarwal and M. Sharir, On the number of congruent simplices in higher dimensions, in preparation.

[2] M. Ajtai, V. Chvátal, M.M. Newborn, and E. Szemerédi, Crossing-free subgraphs, *Ann. Discrete Math.* 12 (1982), 9–12.

[3] N. Alon, H. Last, R. Pinchasi, and M. Sharir, On the complexity of arrangements of circles in the plane, submitted to *Discrete Comput. Geom.*

[4] T.M. Chan, On levels in arrangements of curves, *Proc. 41st IEEE Symp. Found. Comput. Sci.* (2000), to appear.

[5] B. Chazelle and J. Friedman, A deterministic view of random sampling and its use in geometry, *Combinatorica* 10 (1990), 229–249.

[6] K. Clarkson and P. Shor, Applications of random sampling in computational geometry, II, *Discrete Comput. Geom.* 4 (1989), 387–421.

[7] K. Clarkson, H. Edelsbrunner, L. Guibas, M. Sharir, and E. Welzl, Combinatorial complexity bounds for arrangements of curves and spheres, *Discrete Comput. Geom.* 5 (1990), 99–160.

[8] T. Dey and J. Pach, Extremal problems for geometric hypergraphs, *Discrete Comput. Geom.* 19 (1998), 473–484.

[9] H. Edelsbrunner, *Algorithms in Combinatorial Geometry*, Springer Verlag, Heidelberg 1987.

[10] F.T. Leighton, *Complexity Issues in VLSI, Foundations of Computing Series*, MIT Press, Cambridge, Mass., 1983.

[11] J. Matoušek, Efficient partition trees, *Discrete Comput. Geom.* 8 (1992), 315–334.

[12] J. Pach and P.K. Agarwal, *Combinatorial Geometry*, Wiley-Interscience, New York 1995.

[13] J. Pach and M. Sharir, On the number of incidences between points and curves. *Combinatorics, Probability and Computing* 7 (1998), 121–127.

[14] M. Sharir and P.K. Agarwal, *Davenport Schinzel Sequences and Their Geometric Applications*, Cambridge University Press, New York 1995.

[15] L. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Combinatorics, Probability and Computing* 6 (1997), 353–358.

[16] H. Tamaki and T. Tokuyama, How to cut pseudo-parabolas into segments, *Discrete Comput. Geom.* 19 (1998), 265–290.

Appendix A: A Simple Proof of the Old Bound

In this appendix we use Matoušek's partition theorem (Theorem 3.2) to obtain a simple and straightforward proof of the old bound for the maximum number $I(m,n)$ of incidences between m points and n circles in the plane. We emphasize that, except for the use of this tool, the rest of the proof is completely elementary, and does not require any special machinery.

Compact Proof of the Weaker Bound on $I(m,n)$. Let P be a set of m points and C be a set of n circles in the plane. Partition C as in subsection 3.2 into $C_1, \ldots, C_q$. Consider a fixed subset C_i. Let $p \in P$ be a point that is incident to at least two circles in C_i. Then p is a vertex of the arrangement $\mathcal{A}(C_i)$. Clearly, the total number of incidences between the vertices of $\mathcal{A}(C_i)$ and the circles of C_i is $O(|C_i|^2) = O((n/r)^2)$, so the total number of such incidences, summed over all subsets C_i, is $O(r \cdot (n/r)^2) = O(n^2/r)$.

The number of incidences between the circles of C_i and points p that are incident to just one circle of C_i is, in the preceding notation, at most $|P_i| = m_i$; this is because the dual plane of such a point p must *cross* the simplex Δ_i (and not *contain* it), unless $|C_i| = 1$, a case that we may rule out, as above. Hence the overall number of incidences of this second kind is at most $\sum_{i=1}^{q} m_i = O(mr^{2/3})$.

Hence the total number of incidences between P and C is $O(n^2/r + mr^{2/3})$. We choose $r = \min\{n, n^{6/5}/m^{3/5}\}$, and observe that $1 \le r \le n$ when $m \le n^2$, which we can clearly assume (when $m > n^2$ the trivial bound $I(P,C) = O(m)$ holds). This yields the old known bound [7]

$$I(m,n) = O(m^{3/5}n^{4/5} + m + n).$$

□

Remark: Essentially the same proof also works for incidences between m points and n lines. Here the lines are dualized to points in the plane, so Matoušek's construction is done in the plane, and guarantees that a line can cross at most $O(r^{1/2})$ of the relevant simplices. Continuing as above, we conclude that the total number of incidences is $O(n^2/r + mr^{1/2})$, and an appropriate choice of r yields the bound $O(m^{2/3}n^{2/3} + m + n)$.

This approach can also be used to reconstruct other known bounds, such as the bound, given in [13], for the number of incidences between m points and n curves that are graphs of polynomials of any fixed maximal degree s, as well as other instances. (We remark, though, that the bound in [13] holds in more generality. For the approach used here to work, we need to be able to dualize the curves into points in some higher-dimensional space, as was done in subsection 3.4.)

Appendix B: Bypassing the Tamaki-Tokuyama Technique

In this appendix we present an alternative way of obtaining the bound $I(m,n) \approx O(m^{2/3}n^{2/3} + n^{5/3})$ without having to use the result of Tamaki and Tokuyama on cutting circles. In fact, we obtain a bound that depends on X, which is larger than what is obtained in Theorem 3.1, especially when X is subquadratic; when $X = \Theta(n^2)$, it nearly coincides with the preceding bound.

Let P be a set of m points and C a set of n circles in the plane, with $I = I(P,C)$ incidences between them. Let X denote the number of pairs of intersecting circles in C. Arguing as above, we may assume that each circle of C contains at least three points of P. For a circle $c \in C$, let X_c denote the number of other circles that cross c, and let I_c denote the number of points in $P \cap c$. Let C' be the subset of circles $c \in C$ for which $X_c/I_c \le 4X/I$, and let $C'' = C \setminus C'$. We have

$$2X \ge \sum_{c\in C''} X_c > \frac{4X}{I} \sum_{c\in C''} I_c = \frac{4X}{I} I(P, C'').$$

In other words, $I(P,C'') \le I/2$, so it suffices to obtain an upper bound for $I(P,C')$.

Let $c \in C'$. The points of $P \cap c$ partition c into subarcs, which we consider to be closed. The average number of circles that cross such a subarc of c is at most $2X_c/I_c \le 8X/I$. It follows that at least half of these arcs are crossed by at most $16X/I$ circles. We collect all these arcs of the circles of C' into a (plane drawing of a) multigraph G. The number of edges of G is at least $I/4$: We retain at least half of the arcs on each circle of C', and the total number of arcs along these circles is at least $I(P,C') \ge I/2$.

Let $u, v \in P$ be a pair of points connected by many arcs in G. Then these arcs form lenses whose level (in the terminology of Section 2.2) is at most $k = 16X/I$. Hence, using the Clarkson-Shor analysis technique [6] and the fact that the number of lenses of level 0 in such an arrangement is $O(n^{3/2+\varepsilon})$, for any $\varepsilon > 0$ [3], the number of such lenses is

$$O(k^2 \cdot (n/k)^{3/2+\varepsilon}) = O(n^{3/2+\varepsilon}k^{1/2-\varepsilon}) = O(n^{3/2+\varepsilon}X^{1/2-\varepsilon}/I^{1/2-\varepsilon}).$$

Hence, by making that many cuts along the circles of C', we can eliminate all lenses that arise in the graph G.

Arguing as in the earlier analysis, we thus obtain:

$$I = O\left(\frac{n^{3/2+\varepsilon} X^{1/2-\varepsilon}}{I^{1/2-\varepsilon}} + m^{2/3} X^{1/3} \right),$$

implying that

$$I = O\left(n^{(3+\varepsilon)/(3-\varepsilon)} X^{(1-\varepsilon)/(3-\varepsilon)} + m^{2/3} X^{1/3} \right),$$

for any $\varepsilon > 0$. That is, $I(P, C) \approx O((m^{2/3} + n)X^{1/3})$,

In other words, this gives a more direct proof of a bound similar to that in Theorem 3.1. More precisely, the two bounds nearly coincide when $X = \Theta(n^2)$ (with the new derivation yielding a slightly weaker bound), but when X is smaller, the new bound becomes significantly larger. Note, however, that when m is sufficiently large the new bound becomes $O(m^{2/3}X^{1/3})$ and thus coincides with the bound derived earlier. The 'weakness' of the new bound is that one need to set a slightly larger threshold for m to obtain the bound $O(m^{2/3}X^{1/3})$.

Nevertheless, the new derivation completely bypasses the Tamaki-Tokuyama theorem (Theorem 2.3) on cutting circles into 'pseudo-segments'. The dependence on X can be improved as in the proof of Theorem 3.4.

As earlier, the case of pairwise-intersecting circles (where the number of empty lenses is $O(n)$) can also be handled by this approach. The number of relevant lenses and cuts is $O(nk) = O(nX/I) = O(n^3/I)$, yielding $I = O(n^{3/2} + m^{2/3}n^{2/3})$, this time obtaining exactly the same bound as before.

Appendix C: The Complexity of Heavy Pencils

In this appendix we derive an improved bound on the total number of arcs of 'heavy' pencils in an arrangement of n circles.

Specifically, let C be a set of n circles in the plane. A *pencil* π of *weight* $\omega_\pi = j$ corresponds to a pair of points u, v with exactly j circles of C passing through u and v. The pencil itself is the collection of the smaller arcs of those j circles that connect u and v.

Fix an integer k between 1 and n. We seek an upper bound on the sum of the weights of any set Π of pencils in $\mathcal{A}(C)$ so that no two pencils in Π have any pair of overlapping arcs, and so that $\omega_\pi > k$ for each $\pi \in \Pi$. We obtain this bound as follows.

Let $C^{(0)}$ denote the set of centers of the circles of C in the plane. Apply Matoušek's partitioning technique (Theorem 3.2) to $C^{(0)}$ which, for a given parameter $r \le n$ that will be specified shortly, yields a partitioning of $C^{(0)}$ into $q \le 2r$ subsets, call them $C_1^{(0)}, \ldots, C_q^{(0)}$, each consisting of at most n/r centers, and each contained in some triangle (or line segment), so that no line (in the xy-plane) crosses more than $cr^{1/2}$ triangles, for some absolute constant $c > 0$. We choose r so that $cr^{1/2} = \min\{k/2, cn^{1/2}$ and note that we always have $r \le n$. Let C_i denote the subset of C consisting of circles whose centers are in $C_i^{(0)}$, for $i = 1, \ldots, q$.

Apply the result of Tamaki and Tokuyama (Theorem 2.3) to each C_i, thereby cutting the circles of C into

$$O(r) \cdot O((n/r)^{5/3}) = O(n^{5/3}/r^{2/3})$$

arcs, so that any two arcs, that come from circles in the same subset C_i, intersect at most once.

Consider a pencil $\pi \in \Pi$ connecting a pair of points u, v and consisting of $j \geq k$ circles. Let C_i be a subset that contains at least two circles of π. Then all the arcs of π that lie in those circles, with the possible exception of at most one arc, will be cut by the above process, and none of the cutting points lie in more than one arc of one pencil of Π. To estimate the number of subsets C_i that contain only one arc of π we proceed as follows. The perpendicular bisector ℓ_{uv} of u, v passes through the centers of all circles that participate in π, and thus it must either cross or contain all the triangles/segments that contain the above C_i's. If ℓ_{uv} contains the entire C_i then (a) the line ℓ_{uv} is unique for C_i (although the pair u, v need not be unique), and (b) ℓ_{uv} can be associated with at most two arcs on each circle in C_i, for an overall count of at most $2n$ arcs. In all other cases, ℓ_{uv} crosses the triangle/segment containing C_i, and the number of such C_i's is thus at most $cr^{1/2} \leq k/2$. This implies that the total number of arcs of π (excluding those that correspond to full containment of the corresponding C_i) is at most 4 times larger than the number of arcs that have been cut by the above process. Summing this over all pencils of Π, and recalling that no cutting point is 'charged' by more than one pencil, we conclude that the overall weight of the pencils in Π is at most

$$O(n^{5/3}/r^{2/3}) = O(n^{5/3}/k^{4/3} + n).$$

We have thus shown:

Theorem 5.1. *The total number of arcs in a collection of pencils of weights greater than k with no pair of overlapping arcs is $O(n^{5/3}/k^{4/3} + n)$.*

Remark: Theorem 5.1 improves the bound $O(n^2/k^2)$, noted by Székely [15], when $k \leq n^{1/2}$.