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Abstract

We generalize the Hadwiger(-Danzer-Grünbaum-Klee) theorem on line transversals for an unbounded family of compact convex sets to the case of transversal planes of arbitrary dimension. This is the first Helly-type theorem known for transversals of dimension between 1 and $d-1$.

1 Introduction

In 1956 H. Hadwiger proved the following theorem of Helly type for line transversals to a countably infinite collection of convex sets of special form:

Theorem 1 (Hadwiger [6]). *If every $d+1$ members of a countably infinite collection of disjoint congruent convex bodies in $\mathbb{R}^d$ have a line transversal, then the whole collection does.*

Here, the term 'convex body' is used to mean 'compact convex set with interior points.' And a 'line transversal' to a collection of sets is simply a line meeting all of them.

This was a development in the program that had begun with Helly's theorem [8] (giving sufficient conditions for the existence of a point common to a collection of compact convex sets in terms of points common to small subcollections) and which was extended by Vincensini [10] to include the question of the existence of transversal flats of arbitrary dimension.

It was observed by Santaló [9] that there is no Helly-type theorem for transversals of positive dimension to unrestricted collections of convex bodies, but — beginning with Santaló — a number of theorems of Helly type were found for both line transversals and hyperplane transversals to collections restricted by size or shape, such as unit balls, parallelepipeds, translates of a single convex body, etc.; see [3] and [4] for many of these results.

In 1963 L. Danzer, B. Grünbaum, and V. Klee observed that Hadwiger's proof of Theorem 1 works just as well without the assumption that the bodies in question are congruent, as long as their diameters are bounded:

Theorem 2 (Danzer-Grünbaum-Klee [3]). *For a collection of compact convex sets in $\mathbb{R}^d$ whose union is unbounded while the diameters of its members have finite upper bound, if every $d+1$ members have a line transversal, then the whole collection does.*

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The purpose of this paper is to generalize the Hadwiger(-Danzer-Grünbaum-Klee) theorem further to the case of transversals of arbitrary dimension. This is apparently the first known theorem of Helly type concerning transversals of dimension between 1 and $d-1$.

The generalization is presented in Theorem 3, whose special case $k=1$ is simply the original Hadwiger-Danzer-Grünbaum-Klee theorem. We then give, in Proposition 1, necessary and sufficient conditions for a collection of sets to satisfy the hypothesis of Theorem 3 for a particular k.

Our proof follows the general scheme of Hadwiger's proof for the special case of line transversals, with modifications necessitated by the fact that we are dealing with higher-dimensional flats. We generalize the Hadwiger-Danzer-Grünbaum-Klee condition that the union of the sets in the collection be unbounded, replacing it by the condition that their union be unbounded "in k independent directions." And, here as well, the "Helly number" for the existence of k-flat transversals turns out to be $d+1$, just as it is for line transversals.

In the final section of the paper we show that our hypothesis is tight.

2 The theorem

First, some terminology. Suppose $\mathcal{S}$ is a collection of compact convex sets in $\mathbb{R}^d$, all of bounded diameters, and $S_1, S_2, \ldots$ an *unbounded* sequence of members of $\mathcal{S}$. Assume that the origin O is not contained in any S_i. Choose a point $x_i \in S_i$ for each i, and map it to the unit sphere Σ centered at O by the central projection $f : \mathbb{R}^d \setminus O \to \Sigma$. If the points $f(x_1), f(x_2), \ldots$ approach a limit $y \in \Sigma$, we will identify y with the direction Oy, and refer to y as a *limiting direction* of $\mathcal{S}$.

It is clear that any other choice of $x'_i \in S_i$ would lead to the same limiting direction, and that any other choice of origin would as well. We will call the set of these limiting directions the *limiting direction set* of $\mathcal{S}$, $\mathrm{LDS}(\mathcal{S})$. Finally, by $\dim \mathrm{LDS}(\mathcal{S})$ we will mean the dimension of the linear subspace of $\mathbb{R}^d$ spanned by the points of $\mathrm{LDS}(\mathcal{S})$, i.e., of the affine span of O and $\mathrm{LDS}(\mathcal{S})$.

Definition 1. $\mathcal{S}$ is *k-unbounded* if $\dim \mathrm{LDS}(\mathcal{S}) \geq k$.

Lemma 1. *Suppose $\mathcal{S}$ is a collection of compact convex sets in $\mathbb{R}^d$, all of bounded diameters. Then the union of the sets in $\mathcal{S}$ is unbounded if and only if* $\mathrm{LDS}(\mathcal{S}) \neq \emptyset$.

Proof. If $\mathrm{LDS}(\mathcal{S}) \neq \emptyset$, then it follows from the definition that the union of the members of $\mathcal{S}$ is unbounded. The converse is essentially due to Hadwiger [6]; we paraphrase his proof: Since the union of the sets in $\mathcal{S}$ is unbounded, we can find a sequence $S_1, S_2, \ldots \in \mathcal{S}$ whose distances from O approach infinity. Let us assume, without loss of generality, that each S_i already has positive distance from O. Choose $x_i \in S_i$ for each i. By the compactness of Σ, we can find a subsequence of indices i for which the projections $f(x_i)$ converge to a point $y \in \Sigma$. This shows that $\mathrm{LDS}(\mathcal{S}) \neq \emptyset$, i.e., that $\dim \mathrm{LDS}(\mathcal{S}) \geq 1$. □

We can now state the generalized Hadwiger(-Danzer-Grünbaum-Klee) theorem.

Theorem 3. *If $k < d$ and $\mathcal{S}$ is a k-unbounded collection of compact convex sets with bounded diameters in $\mathbb{R}^d$, every $d+1$ members of which have a k-flat transversal, then $\mathcal{S}$ has a k-flat transversal.*

Proof. The case $k=0$ follows from the original Helly theorem for a (possibly infinite) collection of compact convex sets in $\mathbb{R}^d$.

Suppose $k \geq 1$. By Lemma 1, $\mathrm{LDS}(\mathcal{S}) \neq \emptyset$, i.e., $\dim \mathrm{LDS}(\mathcal{S}) \geq 1$. Choose k linearly independent points in $\mathrm{LDS}(\mathcal{S})$, say $y_1, \ldots, y_k$, and, for each i, let $x_{i1}, x_{i2}, \ldots$ be an unbounded sequence of points such that $f(x_{i1}), f(x_{i2}), \ldots$ converges to y_i, and such that each x_{ij} belongs to a set $S_{ij} \in \mathcal{S}$. Let K be the space spanned linearly by $y_1, \ldots, y_k$.

Suppose $T_1, \ldots, T_{d-k+1}$ are any $d-k+1$ members of $\mathcal{S}$. Since they, together with $S_{i1}, \ldots, S_{ik}$, have a k-transversal, it follows that $T_1, \ldots, T_{d-k+1}$ have k-transversals in directions arbitrarily close to K. (This means that the k-transversals, translated to pass through O, meet the sphere Σ in great $(k-1)$-spheres arbitrarily close to the great $(k-1)$-sphere in which K meets Σ.) Hence, by compactness of the members of $\mathcal{S}$, $T_1, \ldots, T_{d-k+1}$ must have a k-transversal *in* the direction K, i.e., parallel to K.

Now project the collection $\mathcal{S}$ along K onto the $(d-k)$-subspace $K^\perp$ orthogonal to K. Then, in $K^\perp$, we have a collection of compact convex sets every $d-k+1$ of which have a common point, so they all do by the form of Helly's theorem that applies to (possibly infinitely many) compact convex sets [8]. Thus there is a k-flat in $\mathbb{R}^d$ that passes through all the members of $\mathcal{S}$. ☐

An equivalent characterization of k-unboundedness is provided by Proposition 1.

Definition 2. If $2 \leq k \leq d$, if $\varepsilon > 0$, and if T is a set of k points on the unit sphere Σ centered at O in $\mathbb{R}^d$, we say that T is *ε-separated* if every great $(k-2)$-sphere on Σ lies more than ε away from at least one point of T. (For the sake of completeness, we will also consider every set consisting of a single point on Σ ε-separated, for any $\varepsilon > 0$.)

Cf. [2], where an analogous concept is defined for a set $T \subset \mathbb{R}^d$.

Recall that the *width* of a set in $\mathbb{R}^d$ is the width of the smallest slab enclosing the set between two parallel hyperplanes.

Proposition 1. *Let $f : \mathbb{R}^d \setminus O \to \Sigma$ be central projection onto the unit sphere Σ centered at O in $\mathbb{R}^d$. Suppose $1 \leq k \leq d-1$. A collection $\mathcal{S}$ of compact convex sets in $\mathbb{R}^d$ is* k-unbounded *if and only if there is a fixed number $\varepsilon > 0$ such that for each $D > 0$, $\mathcal{S}$ contains k members, $S_1, \ldots, S_k$, each at distance at least D from O, such that for any choice of point $x_i \in S_i$ for each $i \geq 1$, the set $\{f(x_1), \ldots, f(x_k)\}$ of projections is ε-separated.*

Proof. Without loss of generality we may assume that $k \geq 2$: If $k = 1$, we already know that $\mathcal{S}$ is k-unbounded if and only if the union of $\mathcal{S}$ is unbounded, and this is clearly equivalent to the condition that for each $D > 0$, $\mathcal{S}$ contains a member at distance at least D from O.

Suppose first that $\mathcal{S}$ satisfies the condition in Definition 2 for a particular $\varepsilon > 0$. By the compactness of Σ, applied one index at a time, we can refine a sequence of ε-separated k-tuples $(f(x_1), \ldots, f(x_k))$ (provided by Definition 2), for which the distance D from O to each x_i approaches infinity, down to a sequence $(f(x_{i1}), \ldots, f(x_{ik}))$ with the following properties:

(i) each x_{ij} is distinct from O and belongs to some $S_{ij} \in \mathcal{S}$;
(ii) the distance from O to x_{ij} (hence to S_{ij}) approaches infinity with j;
(iii) $\lim_{j\to\infty} f(x_{ij}) = y_i$ exists for each $i = 1, \ldots, k$; and
(iv) $x_{1j}, \ldots, x_{kj}$ are ε-separated for each j.

Suppose $\dim \mathrm{LDS}(\mathcal{S})$ were smaller than k. Then some great $(k-2)$-sphere Σ' on Σ would contain $y_1, \ldots, y_k$, hence—for j sufficiently large—$x_{1j}, \ldots, x_{kj}$ would be arbitrarily close to Σ', contrary to the hypothesis.

Conversely, suppose $\mathcal{S}$ is k-unbounded, i.e., $\dim \mathrm{LDS}(\mathcal{S}) \geq k$. Choose linearly independent points $y_1, \ldots, y_k \in \mathrm{LDS}(\mathcal{S})$. Let Σ' be a great $(k-2)$-sphere on Σ that minimizes its maximum distance to $y_1, \ldots, y_k$; Σ' exists by the compactness of Σ. Call this distance δ. Suppose $x_{i1}, x_{i2}, \ldots$ is a sequence of points receding to infinity for which $\lim_{j\to\infty} f(x_{ij}) = y_i$ and for which $x_{ij} \in S_{ij} \in \mathcal{S}$ for each i, j. Then, for j sufficiently large, the subcollection $\{S_{1j}, \ldots, S_{kj}\}$ has the desired property for $\varepsilon = \delta/4$: If there were a great $(k-2)$-sphere at most ε away from all of $f(S_{1j})\ldots,f(S_{kj})$ for every $j > J$, then there would be at most $\leq 2\varepsilon$ away from $f(x_{ij})$ for all i and for all $j > J$, hence at most $3\varepsilon < \delta$ away from $y_1,\ldots,y_k$, contradicting the definition of δ. ☐

3 Are all the hypotheses necessary?

We now show that in the hypothesis of Theorem 3, neither k nor $d+1$ can be reduced without destroying the validity of the theorem.

We first recall that Hadwiger showed, in [6], that his result would no longer hold if he dropped the hypothesis that the sets in question have bounded diameters. Actually, he showed that the hypothesis of congruence in Theorem 1 could not be dropped, but his example consisted of sets of unbounded diameters satisfying the remaining conditions of Theorem 1, and in particular forming a 1-unbounded collection. Thus his example applies equally well to our generalized theorem.

We recall next that Hadwiger showed, by a second example, that his result would no longer hold if he dropped the hypothesis that there were infinitely many sets, even if they were congruent and disjoint from one another: Take four unit disks centered at the vertices of a square and enlarge them until they almost touch. Then it is easy to see that any three will have a line transversal but that all four do not. This also shows that in Theorem 3, the condition that the sets are k-unbounded cannot be dropped.

But what about merely weakening this condition, and assuming only that $\mathcal{S}$ is $(k-1)$-unbounded?

The following proposition shows, for the case $d = 3$, $k = 2$, that this would also be enough to make Theorem 3 fail; the example generalizes.

Proposition 2. *There exists a* 2*-unbounded collection of compact convex sets with bounded diameters in* $\mathbb{R}^3$ *that has no plane transversal, even though every four of its members do.*

Proof. We begin as in one of the Hadwiger examples in [6]: Consider the four (closed) regions inside the disk of radius 1 in the x,y-plane centered at $(0,0,0)$ and outside an inscribed square, and choose a pair of opposite regions from among them. Now rotate that pair of regions about the origin in the x,y-plane through an angle of $\pi/4$ in the positive direction, and enlarge them at the same time so that the radius of the disk becomes 2. Repeat with another rotation through $\pi/4$ and another doubling of the radius, and then once more. The result is shown in Figure 1.

It has the property that any three of the eight (shaded) sets have a line transversal, in fact one passing through the common center, but all eight sets do not [6]. Notice also that some three of the sets have only a discrete set of transversals: those labeled a, b, and c in Figure 1, for example, have only two.

Next, enlarge each of the eight resulting sets very slightly, keeping them pairwise disjoint and without a common transversal, and lift the eight sets, in some chosen order, to a rapidly increasing sequence of heights $z = z_1,\ldots,z_8$. Now lift the eight enlarged sets again to *new* heights $z = z_9,\ldots,z_{16}$, and continue the sequence of liftings, choosing each new height much greater than the ones before. We obtain a countably infinite collection $\mathcal{S}$ of sets at rapidly growing heights $z_1, z_2, \ldots$. To show that every four members of $\mathcal{S}$ have a plane transversal, project the three lowest ones among the four to the x,y-plane, find a line transversal (through $(0,0,0)$) cutting the projections in interior points (this is why we enlarged the original sets!), note that the vertical plane through this line cuts the three lowest sets of the four in interior points also, and then tilt this plane very slightly until it also meets the fourth set; this will be possible if the successive heights $z_1, z_2, \ldots$ have been chosen to increase rapidly enough. The collection $\mathcal{S}$ is clearly 1-unbounded, but not 2-unbounded. Moreover, it has no plane transversal, since any plane cutting all the sets would clearly have to be

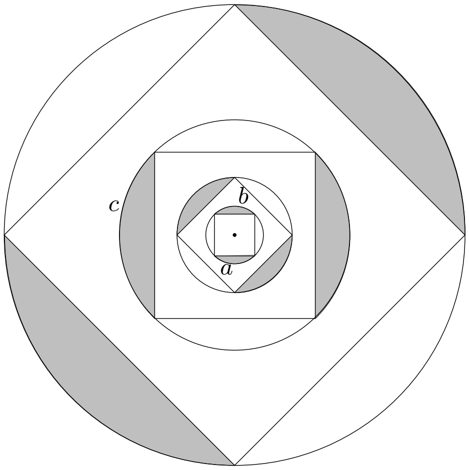

Figure 1: The eight basic sets

vertical, and then its intersection with the x, y-plane would have to be a line cutting all eight of the original regions; no such line exists. ☐

Finally, what about assuming only that every d (rather than $d+1$) members of $\mathcal{S}$ have a k-transversal? Here is a counterexample for the case $d = 3$, $k = 1$; again, it generalizes.

Proposition 3. *There exists a* 1*-unbounded collection of compact sets with bounded diameters in* $\mathbb{R}^3$*, every four members of which have a line transversal, but which itself has none.*

Proof. Consider the three sides, a, b, c, of (say) an equilateral triangle in the x, y-plane, and form the sequence $a_1, b_2, c_3, a_4, b_5, c_6, a_7, \ldots$, where s_n means side s lifted to the height $z = n$; see Figure 2.

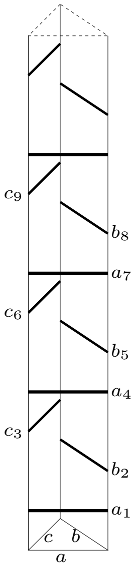

Figure 2: The sequence of segments $a_1, b_2, c_3, \ldots$

This collection of compact convex sets is 1-unbounded, the sets have bounded diameters, and any three of the sets have a line transversal. The last assertion is clear for, say, three sets of the form a_i, a_j, a_k, and also for three sets of the form, say, a_i, a_j, b_k; in each case a suitably chosen vertical line will do. For three sets of the form a_i, b_j, c_k, with, say, $i < j < k$, the line that passes through the endpoints of a_i and c_k whose projection contains b will meet all three segments. Yet the entire collection clearly has no line transversal, since such a line would have to be vertical in order to meet sets that are arbitrarily high, and then its intersection with the x, y-plane would have to be a point lying on all three segments a, b, c, which is impossible. $\square$

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