Small-size ε-Nets for Axis-Parallel Rectangles and Boxes*

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Abstract

We show the existence of ε-nets of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$ for planar point sets and axis-parallel rectangular ranges. The same bound holds for points in the plane with "fat" triangular ranges, and for point sets in $\mathbb{R}^3$ and axis-parallel boxes; these are the first known non-trivial bounds for these range spaces. Our technique also yields improved bounds on the size of ε-nets in the more general context considered by Clarkson and Varadarajan. For example, we show the existence of ε-nets of size $O\left(\frac{1}{\varepsilon}\log\log\log\frac{1}{\varepsilon}\right)$ for the dual range space of "fat" regions and planar point sets (where the regions are the ground objects and the ranges are subsets stabbed by points). Plugging our bounds into the technique of Brönnimann and Goodrich, we obtain improved approximation factors (computable in randomized polynomial time) for the HITTING SET or the SET COVER problems associated with the corresponding range spaces.

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1 Introduction

Since their introduction in 1987 by Haussler and Welzl [HW87] (see also Clarkson [Cla87] and Clarkson and Shor [CS89] for related techniques), ε-nets have become one of the central concepts in computational and combinatorial geometry, and have been used in a variety of applications, such as range searching, geometric partitions, and bounds on curve-point incidences, to name a few; see, e.g., Matoušek [Mat02]. We recall their definition: A *range space* $(X, \mathcal{R})$ is a pair consisting of an underlying universe X of objects, and a certain collection $\mathcal{R} \subseteq 2^X$ of subsets (*ranges*). Of particular interest are range spaces of *finite VC-dimension*; the reader is referred to [HW87] for the exact definition. Informally, it suffices to require that, for any finite subset $P \subset X$, the number of distinct sets $r \cap P$, for $r \in \mathcal{R}$, is $O(|P|^d)$, for some constant d (which is upper bounded by the VC-dimension of $(X, \mathcal{R})$).

Given a range space $(X, \mathcal{R})$, a finite subset $P \subset X$, and a parameter $0 < \varepsilon < 1$, an *ε-net for P and $\mathcal{R}$* is a subset $N \subseteq P$ with the property that any range $r \in \mathcal{R}$ with $|r \cap P| \geq \varepsilon|P|$ contains an element of N. In other words, N is a hitting set for all the "heavy" ranges.

The epsilon-net theorem of Haussler and Welzl asserts that, for any $(X, \mathcal{R})$, P, and ε as above, such that $(X, \mathcal{R})$ has finite VC-dimension d, there exists an ε-net N of size $O\left(\frac{d}{\varepsilon}\log\frac{d}{\varepsilon}\right)$, and that in fact a random sample of P of that size is an ε-net with constant probability. In particular, the size of N is independent of the size of P. The bound on the size of the ε-net was later improved to $O\left(\frac{d}{\varepsilon}\log\frac{1}{\varepsilon}\right)$ by Blumer *et al.* [BEHW89], and then to $(1 + o(1))\frac{d}{\varepsilon}\log\frac{1}{\varepsilon}$ by Komlós, Pach, and Woeginger [KPW92].

In geometric applications, this abstract framework is used as follows. The ground set X is typically a set of simple geometric objects (points, lines, hyperplanes), and the ranges in $\mathcal{R}$ are defined in terms of intersection with (or, for point objects, containment in) simply-shaped regions (halfspaces, balls, simplices, etc.), formally assumed to be regions of *constant descriptive complexity*, meaning that they are semi-algebraic sets defined in terms of a constant number of polynomial equations and inequalities of constant maximum degree. It is known that in such cases the resulting range space $(X, \mathcal{R})$ does have finite VC-dimension (see, e.g., [SA95]).

For example, the main result of our paper concerns the range space in which the objects are points in the plane and the ranges are axis-parallel rectangles; more precisely, each range is the intersection of the ground set with such a rectangle. The *dual* range space in this case is one in which the objects are rectangles and each point p in the plane defines a range which is the subset of the given rectangles that contain p. An ε-net in this case is a subset of the rectangles that covers all the "deep" points.

One of the major questions in the theory of ε-nets, open since their introduction more than 20 years ago, is whether the factor $\log\frac{1}{\varepsilon}$ in the upper bound on their size is really necessary, especially in typical low-dimensional geometric situations. To be precise, in the general abstract context the answer is "yes", as shown by Komlós, Pach, and Woeginger [KPW92], using a randomized construction on abstract hypergraphs (see also [PA95]). However, there is no known lower bound, better than the trivial $\Omega(1/\varepsilon)$, in any "concrete" case, certainly in any geometric situation of the kind mentioned above. The prevailing conjecture is that, at least in these geometric scenarios, there always exists an ε-net of size $O(1/\varepsilon)$ [MSW90].

This "linear" upper bound has indeed been established for a few special cases, such as point objects and halfspace ranges in two and three dimensions, and point objects and disk or pseudo-disk ranges in the plane; see [MSW90, Mat92, CV07, HKSS08, PR08]. Additional progress was made recently. Clarkson and Varadarajan [CV07], essentially adapting Matoušek's technique [Mat92] to their more general setting, have introduced a method for constructing small-size ε-nets in dual range spaces arising in geometric situations where, as above, the ground set is a collection of regions, and each point p determines a range equal to the set of those regions which contain p, and where the combinatorial complexity of the *union* of any finite number r of the regions in the ground set is small, specifically $o(r \log r)$. (The exact condition is slightly more involved—see below.) As a matter of fact, albeit not explicitly presented in this manner, the technique of [CV07] is more general and can also be applied to the primal version of the problem, provided that it satisfies a condition analogous to the one on small union complexity; see later in the paper for more details. More recently, Pyrga and Ray [PR08] have proposed a general abstract scheme for constructing small-size ε-nets in hypergraphs (i.e., range spaces) which satisfy certain properties, and have applied it to the special cases of halfspaces in two and three dimensions, and to several other related scenarios.

The set cover and hitting set problems. Given a range space $(P, \mathcal{R})$, with P and $\mathcal{R}$ finite, the SET COVER problem is to find a minimum-size subcollection $S \subseteq \mathcal{R}$, whose union covers P. A related (dual) problem is the HITTING SET problem, where we want to find a smallest-cardinality subset $H \subseteq P$, with the property that each range $r \in \mathcal{R}$ intersects H. Equivalently, a set cover for $(P, \mathcal{R})$ is a hitting set for the dual range space. The general (primal and dual) problems are NP-hard to solve (even approximately) [GJ79, Kar72], and the simple greedy algorithm yields the (asymptotically) best known approximation factor of $O(1 + \log |P|)$ computable by a polynomial-time algorithm [BGLR93, Fei98]. Most of these problems remain NP-hard even in geometric settings [FG88, FPT81]. However one can attain an improved approximation factor of $O(\log \mathrm{OPT})$ in polynomial time for many of these scenarios, where OPT is the size of the optimal solution. This improvement is based on the technique of Brönnimann and Goodrich [BG95] (see also Clarkson [Cla93]), where the key observation is the relation to ε-nets: The existence of an ε-net of size $O\left(\frac{1}{\varepsilon}\varphi\left(\frac{1}{\varepsilon}\right)\right)$, for any $\varepsilon > 0$, implies that the Brönnimann–Goodrich technique generates, in expected polynomial time, a hitting set (or a set cover) whose size is $O(\mathrm{OPT} \cdot \varphi(\mathrm{OPT}))$.

In other words, improved bounds for the size of ε-nets, in the primal or the dual setting, imply improved approximation factors for the corresponding SET COVER or HITTING SET problems, at least in the context of randomized polynomial-time construction.

Our results. In this paper we begin by considering the cases of point objects and axis-parallel rectangular ranges in the plane, and of point objects and axis-parallel box ranges in three dimensions, and show that both range spaces admit ε-nets of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$, thus significantly improving the standard bound $O\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$. Our technique is similar in spirit to those of Chazelle and Friedman [CF90] and of Clarkson and Varadarajan [CV07], but it differs from them in one key (and fairly simple) idea, which, incidentally, can also be used in the more general context of [CV07] to improve the bounds that are obtained there for the size of the respective ε-nets—see below. An interesting feature of our technique is that it can be extended to points and axis-parallel boxes in *any* dimension, provided that the input points are randomly and uniformly distributed in the unit cube.

We also describe how to construct these ε-nets in randomized expected nearly-linear time. Our results then lead to randomized polynomial-time approximation algorithms for the HITTING SET problem in these two range spaces, involving axis-parallel rectangles and boxes, respectively, which guarantee an approximation factor of $O(\log\log \mathrm{OPT})$.

We then extend our technique to the case of planar point sets and α-*fat* triangles, that is, triangles, each of whose angles is at least α, for some constant $\alpha > 0$ (see [MPSSW94]). In this case too we show the existence of ε-nets of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$, leading to an approximation factor of $O(\log\log \text{OPT})$ for the corresponding HITTING SET problem.

Similarly, we obtain improved bounds for the size of ε-nets in the dual range space, and, consequently, for approximation factors for the corresponding SET COVER problem, in the following cases, all involving points and regions in the plane (refer to Figure 3): (i) *α-fat triangles*. In this case the size of the corresponding ε-net is $O\left(\frac{1}{\varepsilon}\log\log\log\frac{1}{\varepsilon}\right)$, and, as a consequence, the approximation factor for the SET COVER problem becomes $O(\log\log\log \text{OPT})$. (ii) *Locally γ-fat objects*, that is, objects o satisfying the property that, for any disk D whose center lies in o, such that D does not fully contain o in its interior, we have $area(D \sqcap o) \geq \gamma \cdot area(D)$, where $D \sqcap o$ is the connected component of $D \cap o$ that contains the center of D (see [dB08]). If we also assume that each object has a boundary with only $O(1)$ locally x-extreme points, and the boundaries of any pair of input objects intersect in at most s points, for some constant s, then the size of the ε-net is $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$, and the approximation factor for the SET COVER problem is $O(\log\log \text{OPT})$. (iii) *Locally γ-fat objects of (roughly) equal sizes*. Assuming that the objects satisfy the conditions in the previous case, the bound on the size of the ε-net improves to $O\left(\frac{1}{\varepsilon}\log\beta_{s+2}(\frac{1}{\varepsilon})\right)$, where $\beta_t(q) := \lambda_t(q)/q$, and $\lambda_t(q)$ is the (nearly linear) maximum length of Davenport-Schinzel sequences of order t on q symbols (see [SA95]). The corresponding approximation factor becomes $O(\log\beta_{s+2}(\text{OPT}))$. (iv) *Semi-unbounded pseudo-trapezoids*, each consisting of all points lying above or below some x-monotone arc, each pair of which meet at most s times, for s a constant; see Section 3 for a precise definition. In this case the size of the ε-net is, as in the preceding case, $O\left(\frac{1}{\varepsilon}\log\beta_{s+2}(\frac{1}{\varepsilon})\right)$ and the approximation factor is $O(\log\beta_{s+2}(\text{OPT}))$. If the pseudo-trapezoids are also unbounded in the x-direction (so they become "pseudo-halfplanes") these bounds slightly improve to $O\left(\frac{1}{\varepsilon}\log\beta_{s}(\frac{1}{\varepsilon})\right)$ and $O(\log\beta_{s}(\text{OPT}))$, respectively. (v) *Jordan arcs with three intersections per pair*, where each of the actual objects is the region bounded by some Jordan arc which starts and ends on the x-axis (and otherwise lies above it) and by the portion of the x-axis between these endpoints, and each pair of the bounding Jordan arcs intersect at most three times. In this case, assuming that none of the given objects "wiggles" too much (as in case (ii) above), the size of the ε-net is $O\left(\frac{1}{\varepsilon}\log\alpha(\frac{1}{\varepsilon})\right)$, and the approximation factor is $O(\log\alpha(\text{OPT}))$, where $\alpha(\cdot)$ is the (extremely slowly growing) inverse Ackermann function.

Our technique for rectangles—a brief overview. We start with a brief overview of our analysis, in which we assume some familiarity with the earlier papers [CF90, CV07] cited above. Let P be a given set of n points in the plane. We first sketch a somewhat simpler approach that "almost" works—it does not properly address a certain critical technical issue, but captures the essence of our method. We then briefly describe how to modify it so that it does produce ε-nets of the desired size.

Put $r = 1/\varepsilon$. We draw a random sample R of $s \gg r$ points of P (the specific choice of s, made below, is crucial), and make R part of the ε-net to be constructed, so we only need to handle axis-parallel rectangles which contain at least n/r points, but are *R-empty*, i.e., (axis-parallel) rectangles which do not contain any point of R. To "pierce" every such rectangle, we form the subset $\mathcal{M}$ of maximal R-empty rectangles, so that any other R-empty rectangle is contained in one of them. By the standard ε-net theory of [HW87], with high probability each rectangle of $\mathcal{M}$ contains at most $O\left(\frac{n}{s}\log s\right)$ points of P. Moreover, in a sense that we do not make very precise here, the expected number of points of P in such a rectangle is $O(n/s)$. Since $s \gg r$, most rectangles of $\mathcal{M}$ contain fewer than $\varepsilon n = n/r$ points of P, so a rectangle Q with at least n/r points will not fit into any of them, and we can simply ignore them. For each of the relatively few "heavy" rectangles M of $\mathcal{M}$, we apply the resampling technique of [CF90, CV07], and sample a small subset of $O(t\log t)$ points of $M \cap P$, where $t = s|M \cap P|/n$, to serve as a $(1/t)$-net for $M \cap P$. The union of R and all these samples constitutes the desired ε-net; it is fairly easy to show that this is indeed an ε-net.

This approach does not quite work, because the number of maximal R-empty rectangles can be $\Theta(s^2)$ in the worst case (see, e.g., [NLH84]). Using the technique outlined above literally, turns out to yield a bound of $\Theta\left(\frac{1}{\varepsilon^2}\right)$ on the expected size of the ε-net in the worst case, which is of course much too large.

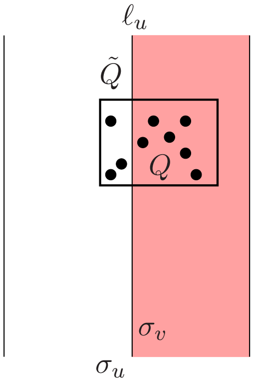


Figure 1: The rectangle Q is anchored at the left entry side ℓ_u of the strip σ_v.

We overcome this issue by modifying the scheme, so that it produces fewer maximal empty rectangles. To do so, we decompose the plane into a binary-tree-like hierarchy of vertical strips. For any rectangle $\tilde{Q}$ which contains at least εn points of P, we find the first (highest in the hierarchy) strip-bounding line which crosses $\tilde{Q}$, take one of its halves, Q, which contains at least $\varepsilon n/2$ points, and consider only such rectangles in the construction of our net. We thus face subproblems, each involving a vertical strip σ and the corresponding subset $P \cap \sigma$ of P, and ranges which are rectangles that are "anchored" at a specific side of σ. The number of maximal R-empty rectangles of this type, within σ, is only *linear* in $|R \cap \sigma|$, leading to an overall collection $\mathcal{M}$ of maximal R-empty rectangles of the new kind, whose size is only $O(s \log r)$.

We now choose $s := cr \log\log r$. Using the so-called Exponential Decay Lemma of [AMS98, CF90], one can show that the expected number of maximal heavy empty rectangles that can contain rectangles Q of the above kind is only *sublinear* in r, which in turn implies that the expected size of the ε-net is dominated by the expected size of R, namely, $O(r \log\log r) = O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$.

Improving the general bounds in [CV07]. Readers familiar with the technique of Clarkson and Varadarajan [CV07] will notice the similarity of our approach to theirs. The key new ingredient is that we use a larger initial sample R, of expected size $\Theta(r \log\log r)$ rather than $O(r)$. The same idea can be applied in the more general context of [CV07], and leads to an improvement of each of their bounds that are super-linear in r. Specifically, Clarkson and Varadarajan consider dual range spaces, and show that if the union complexity of any m of the ranges (i.e., objects in the dual ground set) is $O(m\varphi(m))$, for an appropriate slowly increasing function φ, then there exist ε-nets in such a dual range space of size $O((1/\varepsilon)\varphi(1/\varepsilon))$. Using our approach, we obtain ε-nets of size $O((1/\varepsilon)\log\varphi(1/\varepsilon))$. Moreover, their method yields improved bounds for ε-nets only when $\varphi(m) = o(\log m)$, whereas our method yields improved bounds as long as $\varphi(m) = 2^{o(\log m)}$. The case of rectangles is interesting in this aspect, because, with the addition of the divide-and-conquer decomposition scheme mentioned above, the complexity of the appropriate analog of the union of m dual ranges (which is the number of maximal empty rectangles) is $O(m \log m)$, which is the threshold bound at which the more "naive" approach of [CV07] fails.[1]

We have just learned that very recently Varadarajan [Var08] independently obtained a weaker improvement on the bound of [CV07] for the size of an ε-net in the dual range space of α-fat triangles and planar point sets, using very different methods.

2 Small-size ε-nets for axis-parallel rectangles

Let P be a set of n points in the plane. Put $r := 2/\varepsilon$ and $s := cr \log\log r$, where $c > 1$ is an arbitrary constant. Construct a balanced binary tree $\mathcal{T}$ over the points of P in their x-order, and terminate the tree at the level where the size of each leaf-node is between n/r and $n/(2r)$. By construction, $\mathcal{T}$ has at most $1 + \log r$ levels.

[1] As already noted above, the $\log m$ factor comes from the binary-tree hierarchy—see below for details.

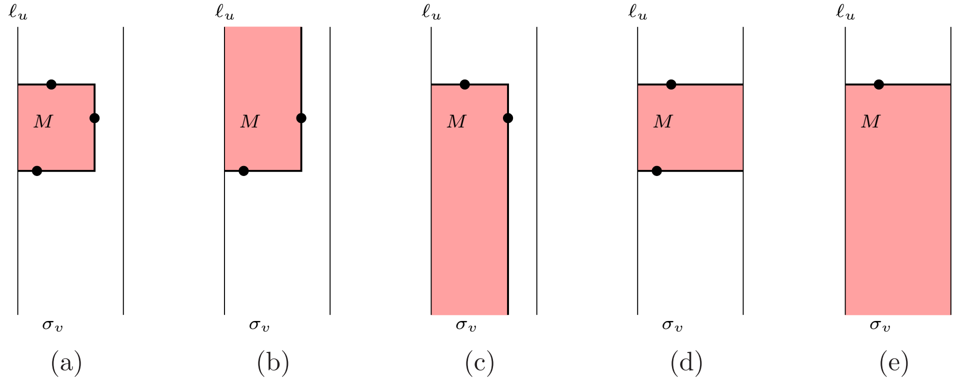


Figure 2: An anchored maximal R-empty rectangle that is determined by three points (a), by a pair of points (b)–(d), or by a single point (e).

Fix a random sample $R \subseteq P$, so that each point $p \in P$ is chosen independently to be included in R with probability $\pi := s/n$; thus the expected size of R is s. The sample R is part of the ε-net N that we are about to construct.

Each node v of $\mathcal{T}$ is associated with a subset P_v of P (resp., R_v of R), consisting of those points of P (resp., of R) stored at the subtree rooted at v. We also associate with v a vertical line ℓ_v which splits P_v into the two subsets P_{v_1}, P_{v_2} associated with the children v_1, v_2 of v. Using the lines ℓ_u, we associate with each node v a strip σ_v, which contains P_v (and R_v), where σ_{root} is the entire plane, and, for a left (resp., right) child node $v \neq$ root of its parent u, σ_v is the left (resp., right) portion of σ_u delimited by ℓ_u. We call ℓ_u the *entry side* of σ_v.

Note that, since the sets P_v are defined ahead of the draw of R, our sampling model guarantees that, for each node v, R_v is an unbiased sample of P_v, drawn from P_v by exactly the same rule, namely, by choosing each point independently with probability π.

Let $\tilde{Q}$ be an axis-parallel rectangle containing at least εn points of P, and let u be the highest node of $\mathcal{T}$ such that ℓ_u crosses $\tilde{Q}$, partitioning it into two parts, one of which necessarily contains at least $\varepsilon n/2 = n/r$ points of P. Denote that portion of $\tilde{Q}$ by Q, and let v be the child of u such that $Q \subseteq \sigma_v$. Q is anchored at the entry side ℓ_u of σ_v; see Figure 1.

If Q contains a point of R, we are done, as $Q \subset \tilde{Q}$ and the goal was to construct a subset of P that meets every rectangle $\tilde{Q}$ containing at least εn points of P. So we may assume that Q does not contain such a point; we then say that Q is *R-empty*; equivalently, Q is *R_v-empty*.

We define, for each node v of $\mathcal{T}$, a set $\mathcal{M}_v$ consisting of all the maximal (open) anchored R_v-empty axis-parallel rectangles contained in σ_v. Without loss of generality, assume that the entry side ℓ_u of σ_v is its left side. In general, a rectangle M in $\mathcal{M}_v$ is determined by three points of R_v, one point lying on each of the three unanchored sides of M (see Figure 2(a)), but $\mathcal{M}_v$ may also contain degenerate rectangles M where some (or all) of these points are missing, in which case M extends as much as possible, within σ_v, in the appropriate direction (upwards, downwards, or to the right). In particular, when $R_v = \emptyset$, there is precisely one maximal R_v-empty rectangle, namely the whole strip; see Figure 2(b)–(e), illustrating some of these cases.

It is easy to show that $|\mathcal{M}_v| = 2r_v + 1$, where $r_v := |R_v|$; we omit the easy proof. (Hints: A point of R_v can lie on the right side of exactly one rectangle in $\mathcal{M}_v$, and all other rectangles in $\mathcal{M}_v$ lie between pairs of points consecutive in the y-order.) It thus follows that the overall number of such maximal empty rectangles $M \in \mathcal{M}_v$, over all nodes v of $\mathcal{T}$ at any fixed level, is $O(|R| + r')$, where r' is the number of nodes at the level, and the total over all levels of $\mathcal{T}$ is $O(r + |R| \log r)$.

Returning now to the anchored rectangle Q and the corresponding node v, we note that Q is contained in at least one rectangle in $\mathcal{M}_v$. Indeed, assuming, as above, that the entry side of σ_v is its left side, expand Q by pushing its right side to the right until it touches a point of R_v or reaches the right side of σ_v, and then push the top and bottom sides until each of them meets a point of R_v or extends to $\pm\infty$. The resulting rectangle belongs to $\mathcal{M}_v$ and encloses Q.

For each node v of $\mathcal{T}$, and each member $M \in \mathcal{M}_v$, define the *weight factor* t_M of M to be $s|M \cap P|/n$. Rectangles M with $t_M < s/r = c \log\log r$ can be ignored, because they contain fewer than n/r points of P, so no anchored rectangle Q, as above, can be completely contained in one of them. By standard ε-net theory [HW87], for each $M \in \mathcal{M}_v$ with $t_M \geq c \log\log r$, there exists a subset $N_M \subseteq M \cap P_v$ of size $c' t_M \log t_M$ that forms a $(1/t_M)$-net for $M \cap P_v$, where c' is another absolute constant.

The final ε-net N is the union of R with the sets N_M, over all the heavy rectangles M (i.e., rectangles with $t_M \geq c \log\log r$) in the respective sets $\mathcal{M}_v$, over all nodes v of $\mathcal{T}$.

N is an ε-net. Since $R \subseteq N$, it suffices to show that for any R-empty rectangle Q, contained in a strip σ_v, anchored at the entry side of σ_v, and containing at least $\varepsilon n/2 = n/r$ points of P (i.e., of P_v), and for any $M \in \mathcal{M}_v$ which contains Q, we have $Q \cap N_M \neq \emptyset$. Therefore,

$$\frac{|Q \cap P|}{|M \cap P|} \geq \frac{n/r}{n t_M/s} = \frac{c \log\log r}{t_M} \geq \frac{1}{t_M}.$$

Since N_M is a $(1/t_M)$-net for $M \cap P$, it follows that $Q \cap N_M \neq \emptyset$, as asserted.

Estimating the size of N. The expected size of N is equal to

$$\mathbf{Exp}\Bigg\{|R| + c' \sum_{v} \sum_{\substack{M \in \mathcal{M}_v \\ t_M \geq c \log\log r}} t_M \log t_M \Bigg\} = cr \log\log r + c' \cdot \mathbf{Exp}\Bigg\{\sum_{v} \sum_{\substack{M \in \mathcal{M}_v \\ t_M \geq c \log\log r}} t_M \log t_M \Bigg\}.$$

We continue the analysis using the notation of [AMS98]. Fix a level i; each node v at this level satisfies $|P_v| = n/2^i$. Let $\mathrm{CT}(R)$ denote the union of the collections $\mathcal{M}_v$, over all nodes v at level i. For a positive parameter t, let $\mathrm{CT}_t(R)$ denote the subset of $\mathrm{CT}(R)$ consisting of those rectangles M with $t_M \geq t$. Let R' denote another random sample of P, where each point $p \in P$ is now chosen, independently, to belong to R' with probability $\pi' := \pi/t$.

Let $\mathcal{C}$ denote the set of all rectangles M, such that M is anchored at the entry side of σ_v, for some node v at level i, and has one point of P on each of its three other sides (the cases of degenerate rectangles, determined by fewer than three points, are treated in a fully analogous manner). For a rectangle $M \in \mathcal{C}$, its *defining set* $D(M)$ is the set of these three points, and its *killing set* $K(M)$ is the set of points of P in the interior of M.

A sufficient condition for the analysis of Agarwal *et al.* [AMS98] to apply is: A rectangle $M \in \mathcal{C}$ belongs to $\mathrm{CT}(R)$ if and only if $D(M) \subseteq R$ and $K(M) \cap R = \emptyset$, which holds by construction in our setting. (We also caution the reader that our sampling model is different from that of [AMS98]—they sample a random subset of a fixed given size uniformly from all such subsets, whereas we independently choose each point of P to belong to the sample. Nevertheless, the lemma, given below, also holds in our model; if at all, the analysis is simpler. For the sake of completeness, we give, in Appendix A.1, a proof of our variant of the lemma.)

Lemma 2.1 (Exponential Decay Lemma; Agarwal *et al.* [AMS98])**.**

$$\mathbf{Exp}\left\{|\,\mathrm{CT}_t(R)|\right\} = O\left(2^{-t}\, \mathbf{Exp}\left\{|\,\mathrm{CT}(R')|\right\}\right).$$

We apply the lemma with $t = c \log\log r$, so $\pi' = \pi/t = r/n$. Recall that $\mathrm{CT}(R')$ is the set of all maximal R'-empty rectangles, anchored at the entry sides of their respective strips σ_v at the fixed level i. Their number is $|\mathrm{CT}(R')| = \sum_v (2r'_v + 1)$, where $R'_v := R' \cap \sigma_v$, and $r'_v := |R'_v|$. Since the sets R'_v at level i are disjoint, $\sum_v r'_v = |R'|$. Hence, since there are at most $2r$ nodes at a fixed level of the tree, we have $|\mathrm{CT}(R')| \le 2|R'| + 2r$. Hence $\mathbf{Exp}\left\{|\mathrm{CT}(R')|\right\} = O(r)$. We thus have $\mathbf{Exp}\left\{|\mathrm{CT}_t(R)|\right\} = O\left(2^{-t}\,\mathbf{Exp}\left\{|\mathrm{CT}(R')|\right\}\right) = O\left(r2^{-c\log\log r}\right) = O\left(r/\log^c r\right)$. Then we obtain (see Appendix A.2 for the easy proof)

$$\mathbf{Exp}\left\{ \sum_{v \text{ at level } i} \sum_{\substack{M \in \mathcal{M}_v \\ t_M \ge t}} t_M \log t_M \right\} = O\left(\frac{r \log\log r \log\log\log r}{\log^c r} \right).$$

Recall again that the analysis so far has been confined to a single level i. Repeating it for each of the $1 + \log r$ levels, we obtain, recalling that $c > 1$, $\mathbf{Exp}\left\{|N|\right\} = O\left(r\log\log r + \frac{r\log\log r \log\log\log r}{\log^{c-1} r} \right) = O(r \log\log r)$. We have thus shown

Theorem 2.2. *For any set P of n points in the plane and a parameter $\varepsilon > 0$, there exists an ε-net of P, of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$, for axis-parallel rectangles.*

Remark: A key ingredient of the analysis is that we have managed to reduce the expected number of R-empty rectangles from $\Theta(s^2)$ to $O(s \log r)$, using a decomposition of the point set into canonical subsets, so that (i) any rectangle $\tilde{Q}$ with at least εn points of P interacts with just *two* subsets (any constant number would do just as well), and (ii) for each canonical subset, the number of maximal R-empty rectangles (now anchored at the entry side of the respective strip and fully contained in that strip) is only linear in the number of sample points in that strip.

In the full version of this paper we present a randomized algorithm for constructing such an ε-net, whose expected running time is $O(n \log n)$; with some extra care, it can be improved to $O(n \log r)$. The algorithm uses fairly standard methods and is omitted.

2.1 Extensions

We have two major extensions of our technique, which we only briefly review here, and postpone the full description to the appendix.

Axis-parallel boxes in three dimensions. Our construction can be extended to the three-dimensional case. To do so, we use a three-level range-tree (see, e.g., [dBCKO08]), which partitions space into box-like cells, each treated as an octant. A "heavy" box B_0 is sent down the tree, and reaches a tertiary node w so that B_0 contains the apex of the octant of w, and the portion B of B_0 within the octant contains at least one eigth of the points in B_0. We then use the fact that, for a given octant σ and a sample R, the number of maximal R-empty axis-parallel boxes which are "anchored" at the apex of σ, is $O(|R \cap \sigma|)$ (see [KRSV07]). These steps allow us to conclude (see Appendix B for the proof)

Theorem 2.3. *For any set P of n points in $\mathbb{R}^3$ and a parameter $\varepsilon > 0$, there exists an ε-net of P, for axis-parallel boxes, of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$.*

Random point sets and boxes in any dimension. When the input set P consists of points in $\mathbb{R}^d$ that are drawn independently and uniformly at random from the unit cube, the expected number of maximal R-empty boxes, for a random sample R of expected size s, as above, is only $O(s \log^{d-1} s)$ [KRSV07]. This leads to a straightforward generalization of our technique (where no partition of space is needed), which yields

Theorem 2.4. *For a random set P of n points in $\mathbb{R}^d$, drawn independently from the uniform distribution on $[0,1]^d$, and a parameter $\varepsilon > 0$, there exists an ε-net of P, for axis-parallel boxes, of expected size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$.*

Fat triangles in the plane. Our next extension is to points in the plane and *α-fat triangles*, where a triangle is α-fat if each of its angles is at least α. Following the analysis of [MPSSW94], it suffices to handle right-angle triangular ranges, each of which has one horizontal edge and one vertical edge, which meet at the lower-left vertex. Here we use a two-level range-tree, where each node of the secondary level is associated with a quadrant. A "heavy" triangle T_0 of the above sort is sent down the tree, and lands at a secondary vertex v so that T_0 is anchored at (contains) the apex of the quadrant of v, and its portion T within the quadrant (which is either a right triangle or a right trapezoid) contains at least one quarter of its points. We then argue that the number of maximal anchored R-empty right triangles or trapezoids within each quadrant σ is only linear in $|R \cap \sigma|$, and thus conclude (see Appendix C for the proof)

Theorem 2.5. *For any set P of n points in the plane, any fixed constant parameter $\alpha > 0$, and a parameter $\varepsilon > 0$, there exists an ε-net of P, for α-fat triangles, of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$, where the constant of proportionality depends on α.*

In each of these cases, there exist randomized algorithms that construct the ε-net in nearly-linear time. Details are omitted in this version.

We next plug the improved bounds on the size of ε-nets into the machinery of Brönnimann and Goodrich [BG95], to obtain improved approximation factors for the corresponding HITTING SET problems (details are given in the full version). We thus obtain:

Corollary 2.6. *There exists a randomized, expected polynomial-time algorithm that, given a set $\mathcal{Q}$ of m axis-parallel rectangles and set P of n points in the plane that hit $\mathcal{Q}$, computes a subset $H \subseteq P$ of $O(\text{OPT} \log\log \text{OPT})$ points that hit $\mathcal{Q}$, where* OPT *is the size of the smallest such set. The same bound holds for the cases of points and axis-parallel boxes in 3-space, random point sets and axis-parallel boxes in any dimension, and planar point sets and α-fat triangles.*

3 Improved bounds for ε-nets for other range spaces

In this section we observe that the technique developed in this paper can be adapted to the scenarios considered by Clarkson and Varadarajan [CV07], and yields improved bounds for the size of ε-nets in many of the cases considered there. As a consequence, using the same implication as in [CV07] (which is based on the technique of Brönnimann and Goodrich [BG95]), but with the improved bounds on the size of ε-nets in the respective range spaces, we obtain approximation algorithms for geometric SET COVER with improved approximation factors.

Rephrasing the notations used in the introduction, we consider the dual range space $\Xi = (\mathcal{C}, \mathcal{Q})$, where the ground set $\mathcal{C}$ is a collection of geometric regions in $\mathbb{R}^d$, and each range in $\mathcal{Q}$ is of the form $Q_x = \{C \in \mathcal{C} \mid x \in C\}$, for some $x \in \mathbb{R}^d$. Clarkson and Varadarajan [CV07] further assume that, for any finite subcollection $\mathcal{C}'$ of m regions of $\mathcal{C}$, the complement of the union of $\mathcal{C}'$ can be decomposed into at most $m\varphi(m)$ *cells* of some simple shape, where $\varphi(m)$ is some slowly increasing function. In addition, each cell in the decomposition is a (possibly unbounded) portion of space that is *defined* by $O(1)$ regions of $\mathcal{C}'$, in the sense that it appears in the decomposition of the complement of the union of those $O(1)$ regions (in particular, the cells of the decomposition do not necessarily have the same shape as the regions of $\mathcal{C}$). In many geometric range spaces of this kind, the cells are those generated by the *vertical decomposition* of the complement of the union [SA95]; see also [AMS98, Cla87, CS89] for a description of this (standard) setup.

Under these assumptions, Clarkson and Varadarajan show that the range space Ξ admits ε-nets of size $O\left(\frac{1}{\varepsilon}\varphi\left(\frac{1}{\varepsilon}\right)\right)$. We obtain the following improvement (see Appendix D for the proof):

Theorem 3.1. *Under the assumptions made above, the range space Ξ admits an ε-net of size $O\left(\frac{1}{\varepsilon}\log\varphi\left(\frac{1}{\varepsilon}\right)\right)$, for any $0 < \varepsilon \le 1$.*

Remark. The bound in the theorem improves upon the general bound $O\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$ when $\varphi(m) = 2^{o(\log m)}$, thus extending the applicability of this technique beyond the "effective range" $\varphi(m) = o(\log m)$, where the original technique of [CV07] yields an improvement.

Several special cases: Bounds for the size of ε-nets and for set cover approximation factors. Theorem 3.1 immediately implies improved bounds on the size of ε-nets for dual range spaces of several classes of regions and points, for which the union complexity (or, rather, the complexity of the decomposition of its complement) is known to be nearly linear. We list some of the standard families with this property, state their union complexity (since we present families of planar regions, the following bounds also apply, with some care, for the complexity of the decomposition of the complement of their union), and the resulting bounds for the size of ε-nets, and for the approximation factors for the corresponding SET COVER problem, computable in (randomized) polynomial time. (For the latter implication to hold, we need (randomized) polynomial-time algorithms for constructing our small-size ε-nets for each of these special classes of regions. Such algorithms follow easily from our constructive proof.)

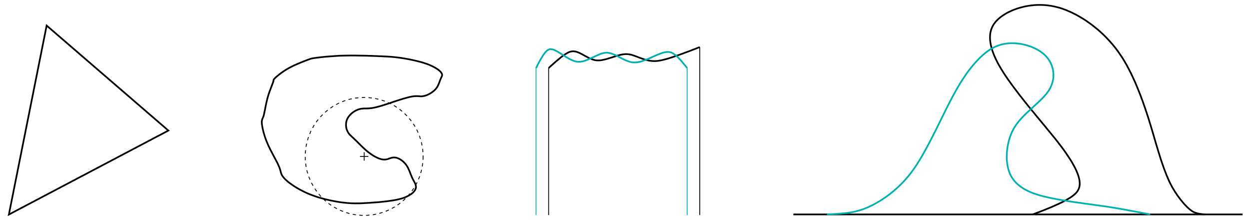
α-fat triangles (Figure 3(a)). Recall that a triangle is α-fat if each of its angles is at least α. The complexity of the union of n such triangles is $O(n \log\log n)$, where the constant of proportionality depends on the fatness factor α [MPSSW94, PT02].

The resulting bound on the size of an ε-net is thus $O\left(\frac{1}{\varepsilon}\log\log\log\frac{1}{\varepsilon}\right)$, and the approximation factor for the corresponding SET COVER problem is $O(\log\log\log \text{OPT})$.

Locally γ-fat objects (Figure 3(b)). These objects were recently introduced by de Berg [dB08]. Given a parameter $0 < \gamma \le 1$, an object o is locally γ-fat if, for any disk D whose center lies in o, such that D does not fully contain o in its interior, we have $area(D \sqcap o) \ge \gamma \cdot area(D)$, where $D \sqcap o$ is the connected component of $D \cap o$ that contains the center of D. We also assume that the boundary of each of the given objects has only $O(1)$ locally x-extreme points, and that the boundaries of any pair of objects intersect in at most s points, for some constant s. It is then shown in [dB08] that the combinatorial complexity of the union of n such objects is $O(\lambda_{s+2}(n)\log^2 n)$, with a constant of proportionality that depends on γ. When the objects have roughly the same size (i.e., the ratio of the diameters of any pair of objects is bounded by some constant), the complexity of the union decreases to $O(\lambda_{s+2}(n))$. Locally γ-fat objects are a generalization of several other previously studied classes of "fat" objects [Ef05, EK99, ES00].

The resulting bounds on the size of an ε-net are thus $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$ for the general case, and $O\left(\frac{1}{\varepsilon}\log\beta_{s+2}\left(\frac{1}{\varepsilon}\right)\right)$ for objects of nearly equal size. The approximation factors for the corresponding SET COVER problems are $O(\log\log \text{OPT})$ and $O(\log\beta_{s+2}(\text{OPT}))$, respectively.

Semi-unbounded pseudo-trapezoids (Figure 3(c)). Here each object is a region of one of the forms $\tau^-_{x_1,x_2,f} = \{(x,y) \mid x_1 \le x \le x_2,\ y \le f(x)\}$, or $\tau^+_{x_1,x_2,f} = \{(x,y) \mid x_1 \le x \le x_2,\ y \ge f(x)\}$, where f is a continuous function. We assume that the graphs of any pair of these functions intersect in at most s points, for some constant s. In this case the complexity of the union of any n such



(a) (b) (c) (d)

Figure 3: The types of regions considered in this section: (a) an α-fat triangle; (b) a locally γ-fat region; (c) semi-unbounded pseudo-trapezoids; and (d) regions bounded by Jordan arcs with three intersections per pair.

objects is $O(\lambda_{s+2}(n))$; see, e.g., [SA95]. If the objects are *pseudo-halfplanes*, that is, $x_1 = -\infty$ and $x_2 = +\infty$ for each object, the bound on the union complexity slightly improves to $O(\lambda_s(n))$.

The resulting bounds on the size of an ε-net are thus $O\left(\frac{1}{\varepsilon}\log\beta_{s+2}\left(\frac{1}{\varepsilon}\right)\right)$ for pseudo-trapezoids, and $O\left(\frac{1}{\varepsilon}\log\beta_s\left(\frac{1}{\varepsilon}\right)\right)$ for pseudo-halfplanes. The approximation factors for the corresponding SET COVER problems are $O(\log\beta_{s+2}(\text{OPT}))$ and $O(\log\beta_s(\text{OPT}))$, respectively.

Jordan arcs with three intersections per pair (Figure 3(d)). Each object is bounded by some Jordan arc which starts and ends on the x-axis but otherwise lies above it, and by the portion of the x-axis between these endpoints, and each pair of the bounding Jordan arcs intersect at most three times. In this case the complexity of the union of any n such objects is $O(\lambda_3(n)) = O(n\alpha(n))$; see [EGH*89]. We also assume that the boundary of each object has only $O(1)$ locally x-extreme points.

The resulting bound on the size of an ε-net is thus $O\left(\frac{1}{\varepsilon}\log\alpha\left(\frac{1}{\varepsilon}\right)\right)$, and the approximation factor for the corresponding SET COVER problem is $O(\log\alpha(\text{OPT}))$.

4 Concluding Remarks and Open Problems

(i) One may consider the dual version of the main problem that we have studied. Namely, we are given a collection $\mathcal{C}$ of n axis-parallel rectangles, and each range is the subset of $\mathcal{C}$ stabbed by some point in the plane. Here too the goal is to show the existence of a small-size ε-net, which is a (small-size) subset $\mathcal{C}' \subseteq \mathcal{C}$ whose union contains all the "deep" points (i.e., points contained in at least εn rectangles of $\mathcal{C}$). So far we do not know how to apply our method to this dual setup. We note that Brönnimann and Lenchner, in their conference paper [BL04], claim, without a proof, the existence of ε-nets for this dual range space, of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$.

(ii) Another challenging open problem is to extend our machinery for axis-parallel boxes to dimension $d \geq 4$. The anchoring trick used for $d = 3$ fails, because the number of maximal R-empty orthants in d-space can be $\Theta\left(|R|^{\lfloor d/2\rfloor}\right)$ [KRSV07]. A modest goal is to construct a *weak ε-net* for this setup (that is, the points in the ε-net are not necessarily chosen from the input set). Another goal is to construct weak ε-nets of size $o\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$ for the (primal) range spaces that we have studied in this paper, most notably for points and axis-parallel rectangles. In fact, it would also be interesting to find a simpler construction that yields weak ε-nets of size $O\left(\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right)$.

(iii) Last but not least, there is the problem of constructing small-size ε-nets for the other primal range spaces considered in Section 3, such as those involving planar point sets and locally γ-fat objects, or semi-unbounded pseudo-trapezoids, with the properties assumed in Section 3.

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A Small-size ε-nets for axis-parallel rectangles

A.1 Exponential Decay Lemma

Proof of Lemma 2.1: For a fixed level i, let T denote the collection of all axis-parallel rectangles which are anchored at the entry side of some strip σ_v at that level, and each of their three other sides contains a point of P_v (or extends all the way to the strip boundary or to $\pm\infty$, as appropriate). Let T_t denote the subset of T consisting of all rectangles with weight factor at least t. We have

$$\textbf{Exp}\left\{|\,\mathrm{CT}_t(R)|\right\} = \sum_{M\in T_t} \textbf{Prob}\left\{M \in \mathrm{CT}(R)\right\}, \tag{1}$$

$$\textbf{Exp}\left\{|\,\mathrm{CT}(R')|\right\} = \sum_{M\in T} \textbf{Prob}\left\{M \in \mathrm{CT}(R')\right\} \geq \sum_{M\in T_t} \textbf{Prob}\left\{M \in \mathrm{CT}(R')\right\}. \tag{2}$$

In view of (1) and (2), it suffices to show that, for each $M \in T_t$,

$$\textbf{Prob}\left\{M \in \mathrm{CT}(R)\right\} = O\left(2^{-t}\right) \cdot \textbf{Prob}\left\{M \in \mathrm{CT}(R')\right\}.$$

Let A_M be the event that $D(M) \subset R$ and $K(M) \cap R = \emptyset$, and let A'_M be the event that $D(M) \subset R'$ and $K(M) \cap R' = \emptyset$. In our setup, the event A_M is exactly the event $M \in \mathrm{CT}(R)$, and the event A'_M is exactly the event $M \in \mathrm{CT}(R')$. Moreover, putting $\delta := |D(M)| \leq 3$, $w := |K(M)|$, we have $\textbf{Prob}\{A_M\} = \pi^\delta(1-\pi)^w$, and $\textbf{Prob}\left\{A'_M\right\} = (\pi')^\delta(1-\pi')^w$. Hence

$$\frac{\textbf{Prob}\left\{M \in \mathrm{CT}(R)\right\}}{\textbf{Prob}\left\{M \in \mathrm{CT}(R')\right\}} = \frac{\textbf{Prob}\left\{A_M\right\}}{\textbf{Prob}\left\{A'_M\right\}} = \frac{\pi^\delta(1-\pi)^w}{(\pi')^\delta(1-\pi')^w} = t^\delta\left(\frac{1-\pi}{1-\pi'}\right)^w,$$

substituting $\pi = s/n$, $\pi' = \pi/t$, $w \geq t \cdot n/s$, the latter expression becomes $O\left(2^{-t}\right)$, which completes the proof of the lemma. □

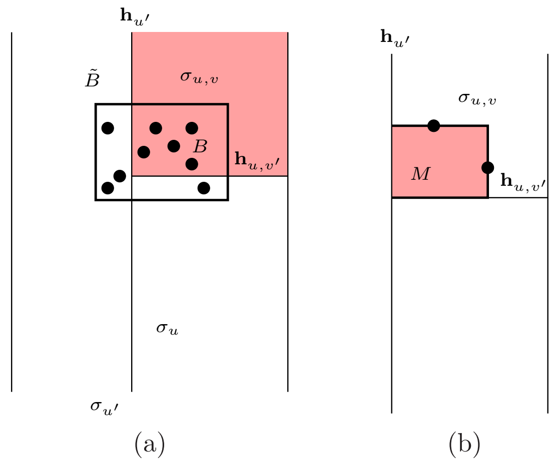

Figure 4: A two-dimensional illustration: (a) The box B is anchored at the (apex of the) quadrant $\sigma_{u,v}$ (octant in 3-space). (b) An anchored box that is determined by a pair of points (a triple in 3-space).

A.2 Estimating the size of N

$$
\begin{aligned}
\mathbf{Exp}\Big\{\sum_{v \text{ at level } i}\ \sum_{\substack{M\in\mathcal{M}_v\\ t_M\geq t}} t_M\log t_M\Big\} &= \mathbf{Exp}\Big\{\sum_{j\geq t}\ \sum_{\substack{M\in\mathrm{CT}(R)\\ t_M=j}} j\log j\Big\}\\
&= \mathbf{Exp}\Big\{\sum_{j\geq t} j\log j\cdot\big(|\,\mathrm{CT}_j(R)|-|\,\mathrm{CT}_{j+1}(R)|\big)\Big\}\\
&= \mathbf{Exp}\Big\{t\log t\cdot|\,\mathrm{CT}_t(R)|\\
&\qquad +\sum_{j>t}\big(j\log j-(j-1)\log(j-1)\big)|\,\mathrm{CT}_j(R)|\Big\}\\
&= O\Big(\frac{r}{\log^c r}(t\log t)+\sum_{j>t}\frac{r}{2^j}\log j\Big)\\
&= O\left(\frac{rt\log t}{\log^c r}\right)=O\left(\frac{r\log\log r\log\log\log r}{\log^c r}\right).
\end{aligned}
$$

B Small-size ε-nets for axis-parallel boxes in three dimensions

We devote this section to prove Theorem 2.3. We now let P be a set of n points in $\mathbb{R}^3$, and put $r := 8/\varepsilon$ and $s := cr\log\log r$, for some fixed constant $c > 3$. We use a similar sampling model as in the two-dimensional problem, in order to generate a random subset $R \subseteq P$ of expected size s.

We next construct a three-level range-tree $\mathcal{T}$, over the points of P (see, e.g., [dBCKO08]), where the points are sorted by their x-coordinates in the primary tree, by their y-coordinates in each secondary tree, and by their z-coordinates in each tertiary tree. We associate with each node u of the primary tree the subset P_u of points that it represents, and a secondary (y-sorted) tree $\mathcal{T}_u$ on P_u. Similarly, with each node v of a secondary tree $\mathcal{T}_u$ we associate the corresponding subset $P_{u,v}$ of P_u and a tertiary (z-sorted) tree $\mathcal{T}_{u,v}$. Finally, each node w of a tertiary tree $\mathcal{T}_{u,v}$ is associated with the corresponding subset $P_{u,v,w}$ of $P_{u,v}$. We construct each of the three levels of $\mathcal{T}$ down to nodes for which the size of their associated subset is between n/r and $n/(8r)$. Clearly, each of the primary, secondary, and tertiary trees has at most $3 + \log r$ levels, and the total number of nodes in the range-tree $\mathcal{T}$ is $O(r \log^2 r)$. Moreover, the sum of the sizes of all the subsets stored at the various nodes is $O(r \log^3 r)$; see, e.g., [dBCKO08] for further details.

Following the notation of Section 2, we associate with each non-leaf node of any subtree an axis-parallel plane which evenly splits the subset stored at the node into the two subsets stored at its children. More specifically, each non-leaf node u of the primary tree stores a plane $\mathbf{h}_u$ orthogonal to the x-axis, each non-leaf node v of a secondary tree $\mathcal{T}_u$ stores a plane $\mathbf{h}_{u,v}$ orthogonal to the y-axis, and each non-leaf node w of a tertiary tree $\mathcal{T}_{u,v}$ stores a plane $\mathbf{h}_{u,v,w}$ orthogonal to the z-axis.

These planes define, for each node w of a tertiary tree $\mathcal{T}_{u,v}$, an octant $\sigma_{u,v,w}$ which is the intersection of three halfspaces $H_u \cap H_{u,v} \cap H_{u,v,w}$, where (i) H_u is the halfspace bounded by $\mathbf{h}_{u'}$ and containing P_u, where u' is the parent of u; (ii) $H_{u,v}$ is the halfspace bounded by $\mathbf{h}_{u,v'}$ and containing $P_{u,v}$, where v' is the parent of v in $\mathcal{T}_u$; and (iii) $H_{u,v,w}$ is the halfspace bounded by $\mathbf{h}_{u,v,w'}$ and containing $P_{u,v,w}$, where w' is the parent of w in $\mathcal{T}_{u,v}$. In what follows we only consider triples (u, v, w) of vertices, each of which has a parent in its respective tree. Thus all three halfspaces are proper, and $\sigma_{u,v,w}$ is a non-degenerate octant.[2]

Let B_0 be an axis-parallel box containing at least εn points of P. Let u' be the highest node in $\mathcal{T}$, so that the plane $\mathbf{h}_{u'}$ meets B_0. This plane partitions B_0 into two portions, one of which, call it B_1, contains at least $\varepsilon n/2$ points of P. Let u be the corresponding child of u' so that H_u contains B_1. Next, let v' be the highest node in $\mathcal{T}_u$, such that $\mathbf{h}_{u,v'}$ meets B_1, partitioning it into two portions, one of which, B_2, contains at least $\varepsilon n/4$ points of P. Let v be the child of v' for which $H_{u,v'}$ contains B_2. Finally, let w' be the highest node in $\mathcal{T}_{u,v}$, such that $\mathbf{h}_{u,v,w'}$ meets B_2, partitioning it into two portions, one of which, B, contains at least $\varepsilon n/8$ points of P. Let w be the child of w' for which $H_{u,v,w'}$ contains B. (Note that u, v, w are well defined, in the sense that each of the sub-boxes is indeed split by a plane associated with a node in the corresponding truncated tree, and does not reach a leaf without being split.)

By construction, B is *anchored* at the resulting octant $\sigma := \sigma_{u,v,w}$, in the sense that the apex o of σ is a vertex of B, and the three facets of B adjacent to o lie on the three respective axis-parallel planar quadrants bounding σ. Moreover, as far as the set $P_{u,v,w}$ is concerned, we can replace B by an octant which is oppositely oriented to σ, and whose apex is the vertex o' of B opposite to o. See Figure 4(a) for an illustration of (the 2-dimensional analog of) this scenario.

For each node w of a tertiary tree $\mathcal{T}_{u,v}$, put $R_{u,v,w} = R \cap \bar{\sigma}_{u,v,w}$ (here it is better to think of $\sigma_{u,v,w}$ as a box), and $r_{u,v,w} = |R_{u,v,w}|$. Let $\mathcal{M}_{u,v,w}$ denote the set of all maximal anchored R-empty (i.e., $R_{u,v,w}$-empty) axis-parallel boxes contained in the octant $\sigma_{u,v,w}$. Since each box $M \in \mathcal{M}_{u,v,w}$ behaves as an octant inside $\sigma_{u,v,w}$, it is determined by at most three points of $R_{u,v,w}$, each lying on a distinct facet of M; see Figure 4(b) for a two-dimensional illustration. The number of such empty boxes (or, rather, octants) is only $O(r_{u,v,w} + 1)$, as shown[3] in [BSTY98, KRSV07]. It thus follows that the overall size of the sets $\mathcal{M}_{u,v,w}$, over all nodes w of all tertiary trees $\mathcal{T}_{u,v}$, is $O(|R| \log^3 r + r \log^2 r)$.

[2]Note, though, that, in general, it is more accurate to regard $\sigma_{u,v,w}$ as a box, or a clipped octant, bounded on the other side also by planes associated with ancestors of u, v, and w. Nevertheless, in most of the following analysis, it suffices to treat $\sigma_{u,v,w}$ as an octant. When we want to emphasize that it is to be treated as a box, we will write $\bar{\sigma}_{u,v,w}$.

[3]In fact, the result in [KRSV07] is more general. It asserts that the number of maximal empty orthants for a set of m points in $\mathbb{R}^d$ is $O(m^{\lfloor d/2 \rfloor})$. It is the non-linearity of this bound for $d \geq 4$ which hampers the extension of our technique to higher dimensions.

We proceed as in the planar case. We make R part of the output ε-net, thereby disposing of any box B_0 whose resulting anchored portion B contains a point of R. For any other box B_0, the corresponding portion B is R-empty, and it is then easy to show that B is contained in at least one maximal R-empty box M in the set $\mathcal{M}_{u,v,w}$ of the corresponding octant $\sigma_{u,v,w}$. Moreover, the weight factor t_M of M, defined as in the planar case, must satisfy $t_M \geq c \log\log r$.

Thus, for each such heavy maximal box M, we take a $(1/t_M)$-net N_M, for the set $P \cap M$, of size $O(t_M \log t_M)$, whose existence is guaranteed by [HW87], and output the union N of R with all the resulting nets N_M. Arguing as in the planar case, it is easy to show that N is indeed an ε-net for P.

We bound the expected size of N using similar analysis steps to those in the planar problem. We define $\mathrm{CT}(R)$ to be the union of all the collections $\mathcal{M}_{u,v,w}$, over all nodes w of all tertiary trees $\mathcal{T}_{u,v}$, appearing in a fixed triple of levels i_1 (primary), i_2 (secondary), and i_3 (tertiary). As before, $\mathrm{CT}_t(R)$ is the subset of $\mathrm{CT}(R)$ consisting of those boxes M with $t_M \geq t$, for any parameter t. It is easy to verify that the Exponential Decay Lemma holds in this scenario as well, and thus

$$\mathbf{Exp}\left\{|\,\mathrm{CT}_t(R)|\right\} = O\left(2^{-t}\,\mathbf{Exp}\left\{|\,\mathrm{CT}(R')|\right\}\right),$$

where R' is another smaller random sample defined as in Section 2. Next, arguing as in the planar problem, we obtain that

$$\mathbf{Exp}\Bigg\{\sum_{v \text{ at levels } i_1,\, i_2,\, i_3} \; \sum_{\substack{M \in \mathcal{M}_v \\ t_M \geq c \log\log r}} t_M \log t_M\Bigg\} = O\left(\frac{r \log\log r \log\log\log r}{\log^c r}\right).$$

Repeating the analysis for each of the $O\left(\log^3 r\right)$ triples i_1, i_2, i_3, we obtain that the expectation of the above sum is $o(r)$, provided $c > 3$, as we indeed assume; thus

$$\mathbf{Exp}\left\{|N|\right\} = \mathbf{Exp}\left\{|R|\right\} + o(r) = O(r \log\log r).$$

C Small-size ε-nets for fat triangles in the plane

In this section we prove Theorem 2.5. We thus have a set P of n points in the plane, and a parameter $\varepsilon > 0$, and our goal is to construct a small-size ε-net $N \subseteq P$, so that any α-fat triangle that contains at least εn points of P contains a point of N.

Passing to semi-canonical triangles. Following the analysis of [MPSSW94], we cover each α-fat triangle T by a triple of "semi-canonical" $(\alpha/2)$-fat triangles, each of which has a pair of edges with orientations taken from a fixed finite set $\mathcal{D}$ of $O(1/\alpha)$ directions, and a third edge that bounds T; see [MPSSW94, Lemma 3.2] and Figure 5(a). Clearly, if T contains at least εn points of P then at least one of the three covering triangles contains at least $\varepsilon n/3$ points of P.

This canonization step yields a constant number ($O(1/\alpha^2)$, to be precise) of subfamilies of $(\alpha/2)$-fat triangles, where the triangles in each subfamily have two edges at fixed orientations (in $\mathcal{D}$), and a third edge whose orientation belongs to a sufficiently small range. Our strategy is thus to construct an $(\varepsilon/3)$-net for P and each of these subfamilies, and the union of all these nets will be an ε-net for P and the family of all α-fat triangles.

Thus, in what follows we focus on a fixed semi-canonical family $\mathcal{F}$. As in [MPSSW94], by applying an appropriate affine transformation, we may assume that each triangle $T \in \mathcal{F}$ is a right triangle with one horizontal edge and one vertical edge, which meet at the lower-left vertex of T; see Figure 5(b).

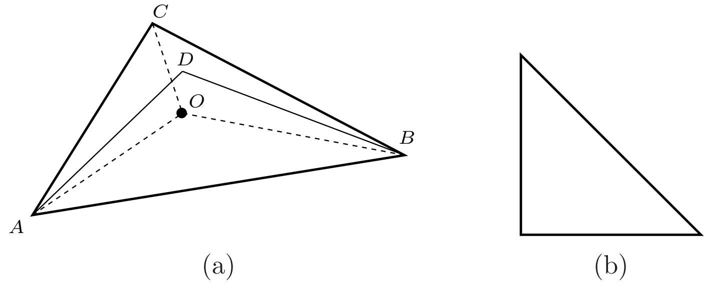

Figure 5: (a) The canonization step. The triangle ABC is covered by three triangles, each of which contains the center O of the inscribed circle of ABC, and has two edge orientations that are taken from a fixed set of directions. Only one of these triangles is depicted in the figure (ABD). (b) A semi-canonical right triangle, after an appropriate affine transformation.

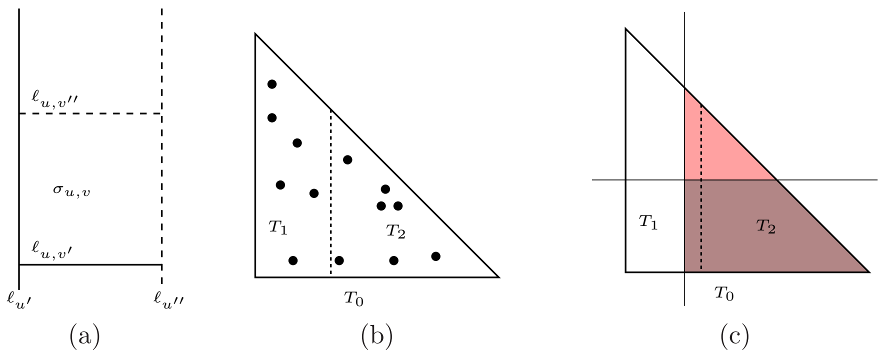

Figure 6: (a) The "quadrant" $\sigma_{u,v}$ is defined by the line splitters $\ell_{u'}$, $\ell_{u,v'}$, but it is also bounded by ancestor splitters $\ell_{u''}$ and $\ell_{u,v''}$. (b) The decomposition of T_0. (c) An anchored triangle T appears as a triangle homothetic to T_0 at the upper-right quadrant, or as a right-angle trapezoid at the lower-right quadrant.

Thus let P and $\mathcal{F}$ be as above, and put $r := 24/\varepsilon$ and $s := cr \log\log r$, for some fixed constant $c > 2$. We use a similar sampling model as in the cases of axis-parallel rectangles and boxes, for drawing a random subset $R \subseteq P$ of expected size s, which becomes part of our ε-net.

We next construct a two-level range-tree $\mathcal{T}$, over the points of P, in an analogous manner to that presented in Appendix B. The points are sorted by their x-coordinates in the primary tree, and by their y-coordinates in each secondary tree, and we construct each of the two levels of $\mathcal{T}$ down to nodes for which the size of their associated subset is between n/r and $n/(4r)$. Following the notation of Appendix B, each node u of the primary tree is associated with the subset P_u of points that it represents, and a secondary (y-sorted) tree $\mathcal{T}_u$ on P_u, and each node v of any secondary tree $\mathcal{T}_u$ is associated with a corresponding subset $P_{u,v}$ of P_u. Each non-leaf node u of the primary tree stores a vertical line "splitter" ℓ_u, and each non-leaf node v of any secondary tree $\mathcal{T}_u$ stores a horizontal line splitter $\ell_{u,v}$. For each such secondary node v of a tree $\mathcal{T}_u$, the lines $\ell_{u'}$ and $\ell_{u,v'}$, where u' is the parent of u in $\mathcal{T}$ and v' is the parent of v in $\mathcal{T}_u$, define a quadrant $\sigma_{u,v}$, which is the intersection of two halfplanes bounded by $\ell_{u'}$ and $\ell_{u,v'}$ and containing $P_{u,v}$. (Technically, similar to the situation in Appendix B, $\sigma_{u,v}$ is a (possibly unbounded) rectangle, where the other vertical and horizontal edges of $\sigma_{u,v}$, if they exist, are portions of respective splitters $\ell_{u''}$, $\ell_{u,v''}$, where u'' is an appropriate ancestor of u' in $\mathcal{T}$ and v'' is an appropriate ancestor of v' in $\mathcal{T}_u$; see Figure 6(a).)

Let T_0 be a right triangle in our semi-canonical family, containing at least $\varepsilon n/3 = 8n/r$ points of P. We first decompose T_0 into two parts, T_1, T_2, by a vertical line, so that T_1 lies to the left of the line and T_2 to its right, and $|T_1 \cap P| \leq |T_2 \cap P| \leq |T_1 \cap P| + 1$. That is, $|T_1 \cap P| \geq 4n/r - 1$

and $|T_2 \cap P| \geq 4n/r$. See Figure 6(b) for an illustration.

As in the case of axis-parallel boxes, we locate the highest node u' in $\mathcal{T}$, so that the line $\ell_{u'}$ meets T_1, thus splitting T_0 into two parts, where the right part is a triangle T', homothetic to T_0 and *fully containing* T_2. In particular, we have $|T' \cap P| \geq 4n/r$. Let u be the right child of u'. We next locate the highest node v' in $\mathcal{T}_u$, such that $\ell_{u,v'}$ meets T'. We focus on the portion T of T' that contains at least $2n/r$ points, and denote by v the child of v' whose corresponding quadrant $\sigma_{u,v}$ contains T.

Brief discussion. (a) Although it may not appear so at first sight, the analysis just given uses also the fact that $|T_1 \cap P|$ is large, to guarantee the existence of the node u in the primary splitting stage: Since we stop the expansion of the primary tree at nodes containing roughly n/r points each, we need to ensure that T_1 contains sufficiently many points of P, or else it would "fall between the cracks" and not be stabbed by any line $\ell_{u'}$. This, however, does not happen due to the way in which T_0 is decomposed,

(b) It is important that T_1 is the portion of T_0 stabbed by $\ell_{u'}$ (and not T_2) because it then ensures that the apex o of $\sigma_{u,v}$ is indeed contained in T_0.

(c) Note that only right children u in the primary tree require the construction of a secondary tree $\mathcal{T}_u$.

The clipped region T is either (a) a triangle, homothetic to T_0, whose right-angle vertex is the apex o of $\sigma_{u,v}$, or (b) a right-angle trapezoid, having o as its top-left vertex, so that its bases are horizontal, its left side is vertical, and its right side is a portion of the hypotenuse of T_0; see Figure 6(c). In both cases we refer to T as being *anchored* at o. Note that in case (a) v is a right child of its parent, representing an upper quadrant, and that in case (b) v is a left child. representing a lower quadrant. Also, in both cases the slope of the slanted edge of T is negative, so in case (b) the slanted edge moves "away" from o, making the lower base of T longer than its upper base.

Recall that we have drawn a "global" random sample R of P. For each node v of each secondary tree $\mathcal{T}_u$, we put $R_{u,v} := R \cap \sigma_{u,v}$ and $r_{u,v} = |R_{u,v}|$. We make R part of the output ε-net N, so if T contains a point of R we are done.

To handle the other case, we define a family $\mathcal{M}_{u,v}$ of maximal anchored $R_{u,v}$-empty regions, with the property that each anchored $R_{u,v}$-empty region T, as above, is covered by at most two regions in $\mathcal{M}_{u,v}$. Each region in $\mathcal{M}_{u,v}$ is either (a) an anchored $R_{u,v}$-empty right triangle whose hypotenuse touches two points of $R_{u,v}$ (that is, it supports an edge of the convex hull of $R_{u,v}$), or (b) an anchored $R_{u,v}$-empty right-angle trapezoid whose slanted side (has negative slope and) touches two points of $R_{u,v}$, and whose unanchored (lower) horizontal base passes through a point of $R_{u,v}$ (which might coincide with one of the two points lying on the slanted edge, i.e., be a vertex of the region), or else lies on the bottom side of the "quadrant" $\sigma_{u,v}$. In each of these cases, the region is clipped within $\sigma_{u,v}$ (when $\sigma_{u,v}$ is defined by three or more splitters). See Figure 7.

In case (a), we also include in $\mathcal{M}_{u,v}$ two axis-parallel rectangles M_1, M_2 anchored at o, so that (i) the right edge of M_1 passes through the leftmost point of $R_{u,v}$ and its top edge lies on the top side of $\sigma_{u,v}$ (if it exists), or else extends to ∞, and (ii) the top edge of M_2 passes through the bottommost point of $R_{u,v}$ and its right edge lies on the right side of $\sigma_{u,v}$ (if it exists), or else extends to ∞. See Figure 8(a). In case (b), we also include in $\mathcal{M}_{u,v}$ axis-parallel rectangles of the following two types: (i) rectangles that are anchored at o, so that each of their right and bottom sides passes through a point of $R_{u,v}$; (ii) rectangles whose left and right sides lie respectively on the left and right sides of $\sigma_{u,v}$ (if the right side exists), and whose top and bottom sides pass through two

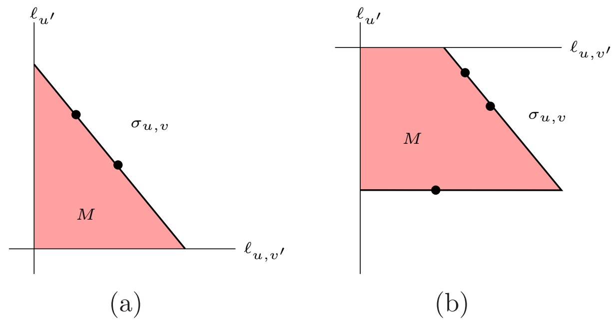

Figure 7: A maximal anchored $R_{u,v}$-empty (a) right triangle, and (b) right trapezoid.

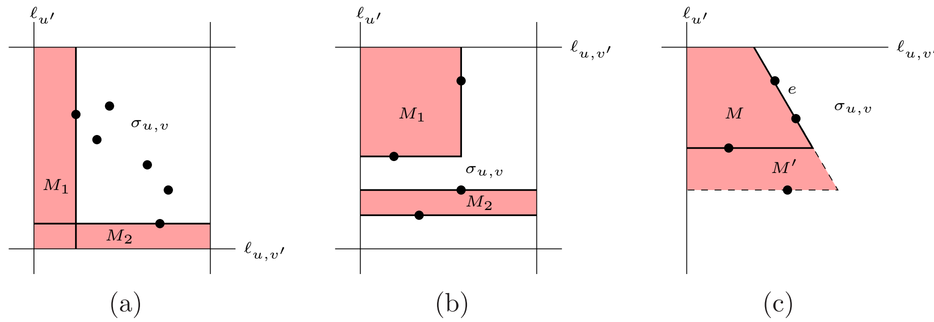

Figure 8: Anchored maximal $R_{u,v}$-empty rectangles (a) for upper quadrants, and (b) for lower quadrants. (c) The right trapezoid M' cannot be empty, if the slanted edge e also belongs to M.

respective points of $R_{u,v}$, necessarily consecutive in the y-order (including two extreme rectangles, above the highest point and below the lowest point). See Figure 8(b). Finally, if $R_{u,v}$ is empty, $\mathcal{M}_{u,v}$ consists of the single region $\sigma_{u,v}$.

We next claim that $|\mathcal{M}_{u,v}| = O(r_{u,v} + 1)$. This is trivial when $R_{u,v} = \emptyset$, so assume that $R_{u,v}$ is nonempty. The claim is then obvious for regions of type (a), because their number is at most two plus the number of edges of the lower-left convex hull of $R_{u,v}$. To bound the number of regions of type (b), sort the points of $R_{u,v}$ in decreasing y-order, and let the sorted sequence be $(q_1, q_2, \ldots, q_{r_{u,v}})$. Put $R^{(j)} = \{q_1, \ldots, q_{j-1}\}$, for $j = 1, \ldots, r_{u,v}$. Let M be a region of type (b) whose lower horizontal base passes through q_j. Then its slanted edge must contain an edge e of the (lower-left) convex hull of $R^{(j)}$. Moreover, if such an M exists then there cannot exist another region M' whose slanted edge contains e and whose lower base passes through any point q_k with $k > j$; see Figure 8(c). Hence the number of regions of type (b) (ignoring the extreme rectangular regions) is upper bounded by the overall number of distinct edges of the "incremental" convex hulls of $R^{(1)}, \ldots, R^{(r_{u,v})}$. The latter number is $O(r_{u,v})$ because every newly added point q_j generates one new edge of the modified hull, possibly deleting several other edges from the hull. (Note that this is exactly the analysis of the classical "Graham scan" convex hull algorithm.) There are only two extreme rectangular ranges of type (a) in $\mathcal{M}_{u,v}$. The number of extreme rectangular ranges of type (b) is easily seen to be $O(r_{u,v})$, using a variant of the analysis in Section 2.

Let $\mathcal{M}$ be the union of all the sets $\mathcal{M}_{u,v}$, over all primary nodes u and all nodes v of the respective secondary trees $\mathcal{T}_u$. Then we have $|\mathcal{M}| = O(|R| \log^2 r + r \log r)$.

We also have the following promised property: Let T be the remaining portion of an initial triangle T_0, and let u and v be the respective primary and secondary nodes for which T is an anchored triangle or trapezoid within $\sigma_{u,v}$, as constructed above. Then, if T is $R_{u,v}$-empty, it is

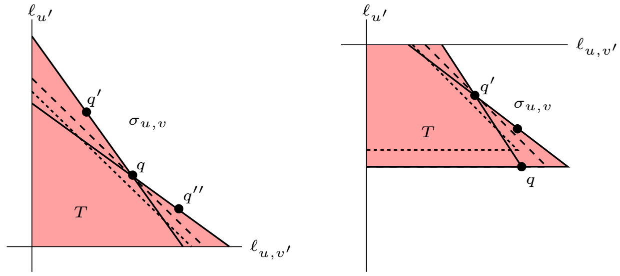

Figure 9: The dotted edges are those of the original triangle or trapezoid T. The dashed edges are the slanted edges of appropriate expansions of the original T. Each such expansion is contained in the union of a pair of regions of $\mathcal{M}_{u,v}$.

contained in the union of at most two regions of $\mathcal{M}_{u,v}$.

Indeed, we may assume that $R_{u,v} \neq \emptyset$. Suppose first that T is a triangle. Translate the hypotenuse of T away from the apex o of $\sigma_{u,v}$, until it passes through a point q of $R_{u,v}$ (necessarily a hull vertex). Then rotate the new hypotenuse about q clockwise (resp., counterclockwise) until it meets a second point q' (resp., q'') of $R_{u,v}$ or becomes vertical (resp., horizontal). The two resulting triangles (or rectangles in the extreme cases) clearly belong to $\mathcal{M}_{u,v}$, and their union covers T. See Figure 9(a).

Suppose next that T is a trapezoid. Expand T homothetically downwards (so that its slanted edge continues to lie on the same line) until its bottom edge hits some point $q = q_j$ of $R_{u,v}$ (that is, in the above notation, q is the jth highest point of $R_{u,v}$), or else reaches the lower boundary of $\sigma_{u,v}$. Then translate the slanted edge of the new trapezoid to the right until it hits a point q' of $R_{u,v}$ (more precisely, of $R^{(j)}$). Finally, rotate the new slanted edge about q' clockwise and counterclockwise until it meets a second point of $R^{(j)}$, or becomes vertical or horizontal; the clockwise rotation may end when it hits $q = q_j$. This yields two trapezoids (or rectangles) of $\mathcal{M}_{u,v}$ whose union covers T. See Figure 9(b). (Note that in both cases, the expansion of T may fall outside $\sigma_{u,v}$. This, however, does not violate our analysis, since in this case T is still contained in the union of at most two regions of $\mathcal{M}_{u,v}$, possibily clipped within $\sigma_{u,v}$.)

By construction, at least one of these two $R_{u,v}$-empty regions must contain at least n/r points of P. The analysis now continues almost verbatim as in Appendix B; that is, for each heavy region $M \in \mathcal{M}_{u,v}$ with weight factor $t_M \geq c \log\log r$, we construct a $(1/t_M)$-net N_M of size $O(t_M \log t_M)$, and output the union N of R with all the resulting nets N_M. The preceding arguments, combined with the analysis in the previous sections, imply that N is indeed an ε-net. Using the Exponential Decay Lemma, which does hold in the present scenario, it can easily be shown that the expected total size of the nets N_M is sublinear in r (for the above choice of c), and thus the expected overall size of the resulting net is $O(r \log\log r)$.

Remark. Once we have restricted our attention to the case of a single semi-canonical family, the remaining analysis does not depend on any assumption concerning the slope of the hypotenuses of the triangular ranges. It thus follows that the bound in Theorem 2.5 also holds for the size of ε-nets for points in the plane and any family of triangular ranges, each of which has a pair of edges at two fixed orientations.

D Improved bounds for ε-nets for other range spaces

Proof of Theorem 3.1: We follow the general approach of Section 2. Here we have a finite subcollection of n elements of $\mathcal{C}$, which, for simplicity, we continue to denote by $\mathcal{C}$. We put $r := 1/\varepsilon$, $s := cr \log \varphi(r)$, and $\pi := s/n$, where $c > 1$ is a constant. We draw a random sample R of regions of $\mathcal{C}$, picking each region, independently, with probability π. We form the union $\mathcal{U}$ of R and decompose its complement into $O(|R|\varphi(|R|))$ simply-shaped regions, each determined by $O(1)$ sets of R; as above, we refer to the regions which form the decomposition as "cells". We define the weight factor t_M of a cell M to be $s|\mathcal{C}_M|/n$, where $\mathcal{C}_M$ is the subcollection of those regions of $\mathcal{C}$ which meet M. By the standard ε-net theory [HW87], or, alternatively, by the Clarkson-Shor technique [Cla87, CS89], it follows that, with high probability, we have $|\mathcal{C}_M| = O\left(\frac{n}{s}\log s\right)$ for each cell M, and, in an informal and imprecise sense, the expected size of $\mathcal{C}_M$, for a cell M, is only $O(n/s)$.[4]

As above, we take each "heavy" cell M, with $t_M \ge c \log \varphi(r)$, and use the standard theory of ε-nets to deduce that there exists a $(1/t_M)$-net N_M for $\mathcal{C}_M$, whose size is $O(t_M \log t_M)$. We output the union of R with all the sets N_M, over all heavy cells M, as the desired $(1/r)$-net (that is, ε-net) N.

Adapting the argument in Section 2, it is straightforward to verify that N is indeed an ε-net. Recall that in this dual context an ε-net is a subset of regions that cover all points that are contained in at least an ε-fraction of the regions. To bound the expected size of N, we follow the same analysis as in Section 2. That is, we apply the Exponential Decay Lemma in this context. Here, for a cell M, its defining set $D(M)$ consists of the $O(1)$ regions that determine M, and its killing set $K(M)$ is the set of all regions in $\mathcal{C}$ that intersect M. In essentially all cases considered in [CV07] and below, the axioms assumed in [AMS98], or their simplified version used in Section 2, hold. We denote by $\mathrm{CT}(R)$ the set of all cells appearing in the decomposition of the complement of the union of a subset R of $\mathcal{C}$, and by $\mathrm{CT}_t(R)$ the subset of $\mathrm{CT}(R)$ consisting of those cells with weight factor at least t.

It thus follows that the Exponential Decay Lemma is applicable in this scenario as well, and it implies that, for any t,

$$\mathbf{Exp}\left\{|\,\mathrm{CT}_t(R)|\right\} = O\left(2^{-t}\,\mathbf{Exp}\left\{|\,\mathrm{CT}(R')|\right\}\right) = O\left(2^{-t}\,\mathbf{Exp}\left\{|R'|\varphi(|R'|)\right\}\right),$$

where R (resp., R') is a random sample in which each region of $\mathcal{C}$ is chosen independently with probability s/n (resp., $s/(tn)$).

To bound the latter expectation, we argue as follows.[5] Let $z := s/t$ denote the expected value of $|R'|$. By Chernoff's bound (see, e.g., [AS92]),

$$\Pr\left\{|R'| \ge \xi z\right\} \le e^{-(\xi-1)^2 z/3},$$

for any $\xi > 1$. Hence, using the sublinearity of φ,

$$\begin{aligned}\mathbf{Exp}\left\{|R'|\varphi(|R'|)\right\} &\le 2z\varphi(2z) + \sum_{j\ge 2} \Pr\left\{|R'| \ge jz\right\}(j+1)z\varphi((j+1)z) \\ &\le z\varphi(z)\cdot\left(4 + \sum_{j\ge 2}(j+1)^2 e^{-(j-1)^2 z/3}\right) = O(z\varphi(z)).\end{aligned}$$

[4] Normally, for these bounds to hold, one needs to consider only those regions of $\mathcal{C}$ which *cross* (i.e., intersect but do not fully contain) M. However, in our case we do not need this distinction: Since each cell M is disjoint from all regions of R, the above analysis also apply to regions of $\mathcal{C}$ that fully contain M.

[5] Here we pay back a little for using the simpler sampling model that we are using.

In particular, for $t = c \log \varphi(r)$, each point is chosen in R' with probability r/n (so $z = r$), and we get

$$\mathbf{Exp}\left\{|\,\mathrm{CT}_t(R)|\right\} = O\left(2^{-c \log \varphi(r)} r\varphi(r)\right) = O\left(\frac{r}{\varphi^{c-1}(r)}\right),$$

which, for $c > 1$, is sublinear in r. For larger values of t, the expectation is $O\left(2^{-t}(s/t)\varphi(s/t)\right)$.

We can now continue with the analysis of Section 2 almost verbatim, arguing that the overall expected size of the subsamples "within" each heavy cell of the complement of the union is sublinear in r, so the expected size of N is dominated by that of R, thus it is $O(r \log \varphi(r))$. $\square$