Functions that are Read-Once on a Subset of their Inputs

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Abstract

A monotone boolean function $f : \{ 0,1 \}^V \rightarrow \{ 0,1 \}$ is *read-once* if f can be expressed as a boolean formula over (AND, OR, NOT) in which every variable in V appears at most once. V. Gurvich (and independently Karchmer, Linial, Newman, Saks, and Wigderson) proved a necessary and sufficient condition for f to be read-once. In this paper we show necessary and sufficient conditions for f to be read-once on a given subset of its inputs. For $Z \subseteq V$, we say that f is *read-once on* Z if f can be expressed as a formula in which every member of Z appears at most once.

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1. Introduction

Let V be a finite set of boolean variables. Let f be a monotone boolean function $f : \{0,1\}^V \rightarrow \{0,1\}$. f is a *read-once* function if f can be expressed as a boolean formula over (AND, OR, NOT) in which every variable in V appears at most once. V. Gurvich (and independently Karchmer, Linial, Newman, Saks, and Wigderson) showed necessary and sufficient conditions for f to be read-once [1,2]. They also developed a combinatorial characterization of read-once functions. In this paper, we show necessary and sufficient conditions for f to be read-once on a given subset of its inputs. For $Z \subseteq V$, we say that f is *read-once on* Z if f can be expressed as a formula in which every member of Z appears at most once.

A subset S of V is called a *minterm* of f if setting all the variables of S to 1 forces the value of f to be 1, and S is minimal with respect to this property. A subset T of V is called a *maxterm* of f if setting all the variables of T to 0 forces the value of f to be 0, and T is minimal with respect to this property. Let $MIN(f)$ denote the set of minterms of f, and let $MAX(f)$ denote the set of maxterms of f. One important property of MIN and MAX is that for all $S \in MIN(f)$, $T \in MAX(f)$, $S \cap T \neq \emptyset$. The proof of this property is by contradiction. If there were an $S' \in MIN(f)$ and a $T' \in MAX(f)$ such that S' and T' were disjoint, then it would be possible to set the variables in S' to 1 (forcing the value of f to be 1) while at the same time setting the variables in T' to 0 (forcing the value of f to be 0).

The following theorem from [1,2] gives a necessary and sufficient condition for f to be read-once.

Theorem 1 [1,2]: If f is a monotone boolean function, then f is read-once iff

$$V - S \in MIN(f), V - T \in MAX(f), |S \cap T| = 1.$$

Is there a similar necessary and sufficient condition on minterms and maxterms for f to be read-once on a subset Z of its inputs? Using a proof very similar to that in [1,2], it is possible to show that the following is a necessary condition for f to be read-once on Z.

Condition 1: $V - S \in MIN(f)$, $V - T \in MAX(f)$, $|S \cap T \cap Z| \leq 1$.

However, Condition 1 not always sufficient. We show another, stronger, condition that is necessary and sufficient in general (Condition 2, defined below). We also address the problem of determining for which cases Condition 1 *is* sufficient for f to be read-once on Z. One motivation for this problem is to generalize Theorem 1. In Theorem 1, $V=Z$. When $V=Z$, $S \cap T \cap Z = S \cap T$, and we know that $S \cap T \neq \emptyset$. Therefore, when $V=Z$, the conditions $|S \cap T \cap Z| \leq 1$ and $|S \cap T| = 1$ are equivalent. We prove that if for all $S \in MIN(f)$, $|S \cap (V-Z)| \leq 1$, then Condition 1 is necessary and sufficient for f to be read-once on Z. Note that the property $|S \cap (V-Z)| \leq 1$ is trivially satisfied whenever $|Z| \geq |V| - 1$. Thus, in particular, Condition 1 is necessary and sufficient when $Z=V$ (Theorem 1), and also when Z contains all but one element of V.

Throughout this paper we rely on the combinatorial characterization of read-once functions described in [1,2]. Before presenting this characterization, we present some background and definitions. We also use the definitions to describe Condition 2.

Without loss of generality, assume that f depends on all variables in V.

We define the *minterm graph* of f to be the graph whose vertex set is V, and whose edge set consists of all pairs of vertices $\{y, z\}$ such that $\{y, z\} \subseteq S$ for some $S \in MIN(f)$. We define the *maxterm graph* of f to be the graph whose vertex set is V, and whose edge set consists of all pairs of vertices $\{y, z\}$ such that $\{y, z\} \subseteq T$ for some $T \in MAX(f)$.

We now present Condition 2.

Condition 2: For all $Z' \subseteq Z$ such that $|Z'| > 1$, either the subgraph of the minterm graph of f induced by Z' is disconnected, or the subgraph of the maxterm graph of f induced by Z' is disconnected (or both).

P_4 is the chordless path on four vertices. A graph is P_4-*free* if it does not contain P_4 as an induced subgraph.

The following result of Seinsche [3] is used in proving the theorem in [1,2].

Lemma 1 [3]: If a graph G is P_4-free and contains more than one vertex, then either G is disconnected, or the complement of G is disconnected.

The combinatorial characterization of read-once functions is described in the following lemma. When we refer to a clique of a graph, we refer to the set of vertices composing the clique.

Lemma 2 [1,2]: If f is a read-once function, then the minterm graph of f is a graph whose maximal cliques are the minterms of f, and whose maximal independent sets are the maxterms of f. The maxterm graph of f is the complement of the minterm graph. Furthermore, the minterm graph is P_4-free, and hence either the minterm graph or the maxterm graph is disconnected.

In Section 2, we prove that Condition 2 is necessary and sufficient for f to be read-once on Z. In Section 3, we give a counterexample that shows that Condition 1 is not always sufficient. The remainder of the paper is devoted to proving that if for all $S \in MIN(f)$, $|S \cap (V-Z)| \leq 1$, then Condition 1 is sufficient for f to be read-once on Z.

2. Condition 2

We show that Condition 2 is necessary and sufficient for f to be read-once on Z.

Theorem 2: Let $f : \{0,1\}^V \rightarrow \{0,1\}$ be a monotone boolean function, and let $Z \subseteq V$. f is read-once on Z iff for all $Z' \subseteq Z$ such that $|Z'| > 1$, either the subgraph of the minterm graph of f induced by Z' is disconnected, or the subgraph of the maxterm graph of f induced by Z' is disconnected (or both).

Proof: ☞ Suppose that f can be expressed as a formula that is read-once on Z. Let F be a binary tree representing such a formula. Without loss of generality assume that all negations in F appear at the leaves. Let Z' be an arbitrary subset of Z. Because each variable in Z' appears only once in F, no variable in Z' is negated in F.

Find the node (call it N) in F closest to the root with the property that both its left and right subtrees contain leaves labeled with elements of Z'. Let F_1 be the formula expressed by the left subtree, and let F_2 be the formula expressed by the right subtree. Let Z_1 be the elements of Z' that appear in F_1, and let Z_2 be the elements of Z' that appear in F_2. Clearly $Z_1 \cup Z_2 = Z'$, and $Z_1 \cap Z_2 = \emptyset$.

Suppose the node N is labeled OR (+). Let $Q \subseteq V$ be a set of variables with the property that setting all the variables of Q to 1 forces the value of F to be 1. If setting the variables of Q to 1 forces the value of F_1 (F_2) to be 1, then clearly $Q \backslash Z_2$ ($Q \backslash Z_1$) will also have the property that setting all of its variables to 1 will force the value of F to be 1. It follows that there is no minterm of f that contains both elements of Z_1 and elements of Z_2. Therefore, if N is labeled OR, the subgraph of the minterm graph of f induced by Z' is disconnected. A similar argument shows that if N is labeled AND, then the subgraph of the maxterm graph of f induced by Z' is disconnected.

☞ Let $G_{Z'}$ denote the subgraph of the minterm graph of f induced by Z', and let $H_{Z'}$ denote the subgraph of the maxterm graph induced by Z'. Suppose that for all $Z' \subseteq Z$ such that $|Z'| > 1$, at least one of $G_{Z'}$ and $H_{Z'}$ is disconnected. This direction of the proof is by induction on the size of Z. The theorem is true for $|Z| = 1$, because every monotone boolean function can be expressed as a formula in which a given variable appears exactly once. Let Z be such that $|Z| > 1$, and assume the theorem holds for all sets of smaller size.

By assumption, either G_Z or H_Z is disconnected. Suppose that G_Z is disconnected. We can partition Z nontrivially into Z_1 and Z_2 such that, in G_Z, no variable in Z_1 is adjacent to a variable in Z_2. Let $\mathbf{S}_1$ be the set of minterms of f containing no variables in Z_2, and let $\mathbf{S}_2$ be the set of minterms of f containing no variables in Z_1. Let f_1 be the monotone function whose minterm set is $\mathbf{S}_1$, and let f_2 be the monotone function whose minterm set is $\mathbf{S}_2$. We will show that there exist formulas F_1 and F_2, expressing f_1 and f_2, that are read-once on Z_1 and Z_2 respectively. We use these formulas to construct a formula for f, namely $F_1 + F_2$, that is read-once on Z. A dual argument holds if H_Z is disconnected.

Let G_{f_1} and H_{f_1} be the minterm and maxterm graphs of f_1. By the definition of f_1, the subgraph of G_{f_1} induced by Z_1 is the subgraph of G induced by Z_1. We now show that the subgraph of H_{f_1} induced by Z_1 is the subgraph of H induced by Z_1.

f_1 is the function obtained from f by restricting all variables in Z_2 to 0. It follows that every maxterm of f_1 is contained in a maxterm of f. Therefore, if $\{a, b\} \in Z_1$ and (a, b) is an edge in H_{f_1}, then

(a, b) is an edge in H.

Now, let T be a maxterm of f. T intersects every minterm of f_1. Therefore, there exists a $T' \subseteq T$ such that T' is a maxterm of f_1. Because T is minimal, $T \cap Z_1 \subseteq T'$. Therefore, if $a, b \in Z_1$ and (a, b) is an edge in H, then (a, b) is an edge of H_{f_1}. It follows that the subgraph of H_{f_1} induced by Z_1 is the subgraph of H induced by Z_1.

By assumption, for all $Z' \subseteq Z$ such that $|Z'| > 1$, at least one of $G_{Z'}$ and $H_{Z'}$ is disconnected. We have just proven that the subgraphs of G_{f_1} and H_{f_1} induced by Z_1 are the subgraphs of G and H induced by Z_1. Therefore, for all $Z' \subseteq Z_1$, either the subgraph of G_{f_1} induced by Z' is disconnected, or the subgraph of H_{f_1} induced by Z' is disconnected (or both). By construction, $|Z_1| < |Z|$. Therefore, by the inductive hypothesis, f_1 can be expressed by a formula that is read-once on Z_1. By the same argument, f_2 can be expressed by a formula that is read-once on Z_2. □

Condition 1 (for all $S \in MIN(f)$, $T \in MAX(f)$, $|S \cap T \cap Z| \leq 1$), is exactly the condition that for all $Z' \subseteq Z$ such that $|Z'| = 2$, the subgraph of the minterm graph of f induced by Z' is disconnected, or the subgraph of the maxterm graph of f induced by Z' is disconnected. Thus, Theorem 2 proves the necessity of Condition 1.

We note that, given the minterm and maxterm graphs of f, it is possible to check in polynomial time whether f is read-once on Z by checking whether Condition 2 holds. Although Condition 2 is stated in terms of all subsets Z' of Z (which seems to imply an exponential procedure), the following procedure is effective and checks only a polynomial number of subsets.

Let G be the minterm graph of f, let H be the maxterm graph, and let G_Z and H_Z be the respective subgraphs induced by Z. Recursively perform the following procedure to test whether f and Z satisfy Condition 2. If $|Z| = 1$, then f and Z trivially satisfy Condition 2. If $|Z| > 1$, then check whether G_Z and H_Z are disconnected. If both are connected, then f and Z do not satisfy Condition 2. Otherwise, at least one of G_Z and H_Z is disconnected. Assume G_Z is disconnected (an analogous procedure works on H_Z if G_Z is connected). For each connected component of G_Z, let Q be the vertices in the connected component, and recursively check whether f and Q satisfy Condition 2. If at least one set Q does not satisfy Condition 2, then f and Z do not satisfy Condition 2, because some subset Z' of Q violates the condition. If all such Q satisfy Condition 2, then Z satisfies Condition 2, because every subset Z' of Z such that $|Z'|>1$ either contains vertices from more than one connected component of G_Z (in which case $G_{Z'}$ is disconnected), or is contained in a tested set Q.

3. Condition 1 is not Sufficient in General

The following counterexample proves that Condition 1 is not sufficient in general.

Let g be the monotone boolean function on $V=\{p,q,r,s,t,x,y\}$ whose minterms are

$$\{\{x,p,q\},\{x,p,r\},\{x,s\},\{y,q\},\{y,r,t\},\{y,s,t\},\{x,y,r\}\}.$$

The maxterms of g are

$$\{\{p,s,y\},\{q,r,s\},\{t,q,x\},\{x,y\}\}.$$

The function g has the property that for all $S\in MIN(g)$, $T\in MAX(g)$, $|S\cap T\cap(X-\{x,y\})|\le 1$. The subgraph of the minterm graph of g induced by $X-\{x,y\}$ is connected. The subgraph of the minterm graph of g induced by $X-\{x,y\}$ is connected. It follows by Theorem 2 that g is not read-once on $X-\{x,y\}$. □

4. Condition 1

In the remainder of this paper, we show that Condition 1 is sufficient if for all $S\in MIN(f)$, $|S\cap(V-Z)|\le 1$.

We say that f *has Property P on Z* if for all $S\in MIN(f)$, $|S\cap(V-Z)|\le 1$, and for all $S\in MIN(f)$, $T\in MAX(f)$, $|S\cap T\cap Z|\le 1$.

Let $G(f,Z)$ be the subgraph of the minterm graph of f induced by the vertices in Z. Let $H(f,Z)$ be the subgraph of the maxterm graph of f induced by the vertices in Z.

In the main part of our proof, we show that if f has Property P on Z, and if $|Z|>1$, then at least one of $G(f,Z)$ and $H(f,Z)$ is disconnected. In the following lemma, we show that if the statement above is true, it follows that if f has Property P on Z, then f is read-once on Z.

Lemma 3: If statement (1) below is true, then statement (2) is also true.

(1) If f has Property P on Z and $|Z|>1$, then at least one of $G(f,Z)$ and $H(f,Z)$ is disconnected.

(2) If f has Property P on Z, then f is read-once on Z.

Proof: Assume statement (1) is true. We prove statement (2) by induction on $|Z|$.

If $|Z|=1$, then f is read-once on Z because every monotone boolean function can be expressed as a formula in which a given input variable appears exactly once. Let $i>1$. Assume that (2) holds when $|Z|<i$.

Let f have Property P on Z, and let $|Z|=i$. By (1), at least one of $G(f,Z)$ and $H(f,Z)$ is disconnected. Suppose $G(f,Z)$ is disconnected. Then Z can be partitioned into two disjoint subsets Z_1 and Z_2 such that $G(f,Z)$ contains no edges between elements of Z_1 and elements of Z_2. As in the proof of Theorem 2, let $\mathbf{S}_1$ be the set of minterms of f containing no variables in Z_2, and let $\mathbf{S}_2$ be the set of minterms of f containing no variables in Z_1. Let f_1 be the monotone function whose minterm set is $\mathbf{S}_1$, and let f_2 be the monotone function whose minterm set is $\mathbf{S}_2$. f_1 is the function obtained from f by restricting all variables in Z_2 to 0. f_2 is the function obtained from f by restricting all variables in Z_1 to 0. By the definition of f, $V-S\in MIN(f), T\in MAX(f)$, $|S\cap T\cap Z|\le 1$. It follows easily that $V-S'\in MIN(f_1)$, $T'\in MAX(f_1)$, $|S'\cap T'\cap Z_1|\le 1$, and $V-S''\in MIN(f_2)$, $T''\in MAX(f_2)$, $|S''\cap T''\cap Z_2|\le 1$. Therefore, f_1 and f_2 have Property P on Z_1 and Z_2 respectively. By the inductive assumption, f_1 can be written as a formula F_1 that is read-once on Z_1, and f_2 can be written as a formula F_2 that is read-once on Z_2. The formula F_1+F_2 is then a formula for f that is read-once on Z. The dual argument holds if $H(f,Z)$ is disconnected. □

Fix f to be an arbitrary monotone boolean function that has Property P on Z. Let $G=G(f,Z)$, and let

$H=H(f,Z)$. We now dedicate ourselves to proving that at least one of G and H is disconnected.

4.1. Preliminaries

Let $W=V-Z$. For $Z' \subseteq Z$, define $G_{Z'}$ to be the subgraph of G induced by Z'. For $W' \subseteq W$, let $f_{W'}$ be the function obtained from f by setting all variables in W' to one, and all variables in $W-W'$ to zero. Let $v(W')$ be the variables on which $f_{W'}$ depends. It follows directly from the definitions that for $W' \subseteq W$, if

$$\alpha=\{ S \setminus W \mid S \in MIN(f) \text{ and } S \cap (W-W')=\emptyset \}$$

then

$$MIN(f_{W'})=\{ S' \mid S' \in \alpha, \text{ and } \not\exists\, S'' \in \alpha \text{ such that } S'' \subset S' \}$$

To simplify our proof, we would like to be able to assume that for all $W' \subseteq W$ such that $W' \neq \emptyset$, $v(W')$ is not empty. Let $W_0=\{ w \in W \mid v(\{ w \})=\emptyset \}$. Suppose $W_0 \neq \emptyset$. Consider the function $f' : \{ 0,1 \}^{V-W_0} \rightarrow \{ 0,1 \}$ produced by setting all elements in W_0 to zero. f' has Property P on Z. If it is possible to produce a read-once formula for f', then a read-once formula for f can be easily produced by appending $+w_1+w_2 \cdots +w_q$ to the formula for f, where $w_1,w_2,\ldots,w_q$ are the members of W_0. Therefore, we can assume that for all $w \in W$, $v(\{ w \}) \neq \emptyset$. It follows that for all $W' \subseteq W$ such that $W' \neq \emptyset$, $v(W') \neq \emptyset$.

The following lemma is fundamental to our proof.

Lemma 4: For all $W' \subseteq W$, $G_{v(W')}$ is the minterm graph of $f_{W'}$, and $H_{v(W')}$ is the maxterm graph of $f_{W'}$. $f_{W'}$ is read-once. $G_{v(W')}$ and $H_{v(W')}$ are P_4-free graphs, and $G_{v(W')}$ is the complement of $H_{v(W')}$. The maximal cliques of $G_{v(W')}$ and $H_{v(W')}$ are respectively the minterms and maxterms of $f_{W'}$.

Proof: If S' is a minterm of $f_{W'}$, then there exists a minterm S of f such that $S' \subseteq S$. If T' is a maxterm of $f_{W'}$, then there exists a maxterm T of f such that $T' \subseteq T$. By the definition of f, $|S \cap T \cap Z| \leq 1$. S' and T' contain no elements of W, so $S' \cap T' \subseteq S \cap T \cap Z$. Therefore, $|S' \cap T'| \leq 1$. Every minterm and maxterm of a monotone function have a non-empty intersection, and so $|S' \cap T'| = 1$. By Theorem 1, $f_{W'}$ is a read-once function.

If $v_1, v_2 \in S'$, then (v_1, v_2) is an edge in G. Therefore, if (v_1, v_2) is an edge in the minterm graph of $f_{W'}$, then it is an edge in $G_{v(W')}$.

Suppose v_1 and v_2 are not adjacent in the minterm graph of $f_{W'}$. Because $f_{W'}$ is a read-once function, v_1 and v_2 are adjacent in the maxterm graph of $f_{W'}$. It follows that there exists a maxterm T' of $f_{W'}$ such that $\{v_1, v_2\} \subseteq T'$. There also exists a maxterm T of f such that $T' \subseteq T$. For all minterms S of f, $|S \cap T \cap Z| \leq 1$. Therefore, there is no minterm of f containing both v_1 and v_2, and hence v_1 and v_2 are not adjacent in G. Thus $G_{v(W')}$ is the minterm graph of $f_{W'}$. The proof that $H_{v(W')}$ is the maxterm graph of $f_{W'}$ follows from a dual argument. The remainder of the lemma is a direct consequence of the combinatorial characterization of read-once functions in Lemma 2. □

4.2. Definitions and a Proof Outline

The proof that at least one of G and H is disconnected is by induction on a sequence of nested subgraphs of G, and a sequence of nested subgraphs of H. The sequences are defined as follows.

Let $m = 2^{|W|}$.

Choose an indexing, $W_1, W_2, \ldots, W_m$, of the subsets of W with the property that if $W_i \subset W_j$, then $i > j$.

Note that $W_1 = W$, and $W_m = \emptyset$.

Let $Z_i = \bigcup_{j=1}^{i} v(W_j)$. Let $G_i = G_{Z_i}$. Let $\tilde{Z}_i = Z_i - Z_{i-1}$.

$G_1 = G_{Z_1} = G_{v(W)}$, $G_m = G$, and $G_1, G_2, \ldots, G_m$ form a nested sequence of subgraphs of G.

Similarly, let $H_i = H_{Z_i}$. $H_1 = H_{Z_1} = H_{v(W)}$, $H_m = H$, and $H_1, H_2, \ldots, H_m$ form a nested sequence of subgraphs of H.

Lemmas 5 through 11 (which appear below) together compose an inductive proof that for all i such that

$1 \le i \le 2|W|$, at least one of G_i and H_i is disconnected. In particular, then, at least one of G and H is disconnected.

The following is an outline of the inductive proof. The base case of the induction is easy. $G_1 = G_{v(W)}$. It follows directly from Lemmas 1 and 4 that G_1 and H_1 are P_4-free, and hence at least one of them is disconnected (In fact, exactly one of them is disconnected). The difficulty in performing the inductive step is that for arbitrary i, G_i and H_i are not necessarily P_4-free. In fact, they may have very complex structures. Fortunately, though, Lemma 4 tells us that G_i and H_i are made up of P_4-free subgraphs. Moreover, as we will see in Lemma 5, it is possible to prove nice properties about pairs of these P_4-free subgraphs. The attack we use in our proof is to reduce problems involving the structure of G_i and H_i to problems involving the structure of one or two P_4-free subgraphs of G_i and H_i.

The inductive step of the proof is essentially broken into three cases:

(1) G_{i-1} is disconnected and G_i is disconnected.

(2) H_{i-1} is disconnected.

(3) G_{i-1} is disconnected and G_i is connected.

Case (1) is trivial, because the inductive condition is satisfied in this case.

In Lemma 9, we prove that if H_{i-1} is disconnected, then H_i is disconnected. This takes care of Case (2).

In Lemma 7, we prove that in Case (3), $G_{v(W_i)}$ is connected, and $G_{v(W_i) \cap Z_{i-1}}$ is disconnected. Lemma 7 is used to prove Lemma 10. Lemma 10 says that in Case (3), H_i is disconnected. The proof of Lemma 10 is roughly as follows. By Lemmas 1 and 4, if $G_{v(W_i)}$ is connected, then $H_{v(W_i)}$ is disconnected. Also by Lemma 4, $H_{v(W_i)}$ is the complement of $G_{v(W_i)}$. It therefore follows from Lemma 7 that in Case (3), $H_{v(W_i)}$ is disconnected, and $H_{v(W_i) \cap Z_{i-1}}$ is connected (because the complement of a disconnected graph is a connected graph). Thus in Case (3), $H_{v(W_i)}$ is disconnected, and contains a connected component C consisting wholly of vertices in $Z{\sim}i$. We show that C must also be a connected component of H_i, and hence H_i is disconnected.

Lemmas 5, 6, and 8, are technical lemmas we use in proving Lemmas 7, 9, and 10.

4.3. Properties of the P_4-free Subgraphs of G and H.

Lemma 5: The following properties hold:

1) Every vertex in G is either in Z_1, or adjacent in G to a vertex in Z_1.

2) If $W_k \subseteq W_j$, and Q is a maximal clique of $G_{v(W_k)}$, then $Q \cap v(W_j)$ is a maximal clique of $G_{v(W_j)}$.

3) If $W_k \subseteq W_j$, and Q is a maximal clique of $H_{v(W_j)}$, then $Q \cap v(W_k)$ is a maximal clique of $H_{v(W_k)}$.

4) For all $j,k \in \{1,...,m\}$, if Q is a maximal clique of $G_{v(W_k)}$, and Q is not a maximal clique of $G_{v(W_k \cup W_j)}$, then $Q \cap v(W_j)$ is a maximal clique of $G_{v(W_j)}$.

5) If $W_j \subset W_k \subset W_l$ then $(v(W_j) \setminus v(W_k)) \cap v(W_l) = \emptyset$.

6) For all $j,k \in \{1,...,m\}$, $v(W_j) \cap v(W_k) \subseteq v(W_j \cup W_k)$.

7) If $Z', Z'' \subseteq v(W_j)$ such that $Z' \cap Z'' = \emptyset$, $G_{Z'}$ connected, $G_{Z''}$ connected, $G_{v(W_j)}$ contains no edges between members of Z' and members of Z'', and there exist $z' \in Z'$, $z'' \in Z''$, $z \in v(W_j) - (Z' \cup Z'')$ such that $z'zz''$ is a path in G, then z is adjacent in G to all vertices in Z' (and symmetrically to all vertices in Z''). The equivalent statement is also true for H.

8) If $W_k \subseteq W_j$, and $C \subseteq v(W_j)$ such that G_C is connected, then $G_{v(W_k) \cap C}$ is either empty or connected.

Proof:

1) $Z_1 = v(W)$. Let z be a vertex of G. By definition, $G = G(f, Z)$, so $z \in Z$. Then $z \in v(\{w\})$ for some $w \in W$. z is contained in some maximal clique Q of $G_{v(\{w\})}$. By Lemma 4, Q is a minterm of $f_{\{w\}}$. Because $v(W_i) \neq \emptyset$ for all $W_i \subseteq W$, there exists a (non-empty) minterm Q' of f_W such that

$Q' \subseteq Q$. $Q' \subseteq v(W)$. Therefore, either $z \in v(W)$ or z is adjacent to a member of $v(W)$.

2) By Lemma 4, Q is a minterm of f_{W_k}. Because $W_k \subseteq W_j$, f_{W_j} contains a minterm Q' such that $Q' \subseteq Q$. Q' is a maximal clique of $G_{v(W_j)}$, and $Q' \subseteq Q \cap v(W_j) \subseteq v(W_j)$, hence $Q' = Q \cap v(W_j)$.

3) Dual of Property 2.

4) For all $S \in MIN(f)$, $|S \cap W| \leq 1$, so if S' is a minterm of $f_{W_k \cup W_j}$, then S' is a minterm of f_{W_k} or f_{W_j}.

By Lemma 4, Q is a minterm of f_{W_k}. Q is not a minterm of $f_{W_k \cup W_j}$ because Q is not a maximal clique of $G_{v(W_k \cup W_j)}$. Therefore, $f_{W_k \cup W_j}$ must contain a minterm Q' such that $Q' \subset Q$. Furthermore, Q' must be a minterm of f_{W_j}, and hence Q' is a maximal clique of $G_{v(W_j)}$. $Q' \subseteq Q \cap v(W_j) \subseteq v(W_j)$ so $Q' = Q \cap v(W_j)$.

5) Suppose there exists a z such that $z \in (v(W_j) \backslash v(W_k)) \cap v(W_l)$. Let Q be a maximal clique of $G_{v(W_j)}$ containing z. By Property 2, $Q \cap v(W_k)$ is a maximal clique of $G_{v(W_k)}$, and $Q \cap v(W_k) \cap v(W_l)$ is a maximal clique of $G_{v(W_l)}$. But $z \in G_{v(W_l)}$, $z \notin Q \cap v(W_k) \cap v(W_l)$, and $(Q \cap v(W_k) \cap v(W_l)) \cup \{z\}$ is a clique of $G_{v(W_l)}$. This contradicts the maximality of $Q \cap v(W_k) \cap v(W_l)$ in $G_{v(W_l)}$.

6) Let $z \in v(W_j) \cap v(W_k)$. Let Q be a maximal clique of $G_{v(W_k)}$ containing z. By Property 4, either Q is a maximal clique of $G_{v(W_j \cup W_k)}$ (and then $z \in v(W_j \cup W_k)$), or $Q \cap v(W_j)$ is a maximal clique of $G_{v(W_j)}$. If $Q \cap v(W_j)$ is a maximal clique of $G_{v(W_j)}$, then $Q \cap v(W_j)$ is a minterm of $f_{W_j \cup W_k}$, and $z \in v(W_j \cup W_k)$.

7) Suppose z is not adjacent to all vertices in Z'. Then, because $G_{Z'}$ is connected, there exist $z_1, z_2 \in Z'$ such that z_1 and z_2 are adjacent, z is adjacent to z_1 and z is not adjacent to z_2. Then $z_2 z_1 z z''$ is a chordless path in $G_{v(W_j)}$, because G contains no edges between members of Z' and members of Z''. Thus $G_{v(W_j)}$ contains a subgraph isomorphic to P_4. But, by Lemma 4, $G_{v(W_j)}$ is P_4-free. Contradiction. The proof for H is the same.

8) Suppose $G_{v(W_k)\cap C}$ is disconnected. Then there exist $z_1, z_2 \in C \cap v(W_k) \subseteq v(W_k) \cap v(W_j)$ such that z_1 and z_2 are contained in a single connected component of $G_{v(W_j)}$, but in two distinct connected components of $G_{v(W_k)\cap v(W_j)}$. z_1 and z_2 are not adjacent in G. They are connected by a path in $G_{v(W_j)}$. By Lemma 4, $G_{v(W_j)}$ is a P_4-free graph. Therefore, there exists a vertex z' in $v(W_j) \backslash v(W_k)$ such that z' is adjacent to z_1 and z_2. Let C_1 and C_2 be the connected components of $G_{v(W_k)\cap v(W_j)}$ containing z_1 and z_2 respectively. By Property 7, in $G_{v(W_j)}$, z' is adjacent to all vertices in C_1 and C_2.

Let Q' be a maximal clique of $G_{v(W_k)}$ containing z_1. $W_k \subseteq W_j$, so by Property 2, $Q' \cap v(W_j)$ is a maximal clique of $G_{v(W_j)}$. $Q' \cap v(W_j)$ contains z_1, and therefore $Q' \cap v(W_j) \subseteq C_1$. z' is adjacent to all vertices in C', and $z' \notin C_1$. $(Q' \cap v(W_j)) \cup \{z'\}$ is a clique of $G_{v(W_j)}$, contradicting the maximality of $Q' \cap v(W_j)$.

Therefore $G_{C\cap v(W_k)}$ is either empty or connected. □

4.4. The Inductive Proof

In this section we prove that if at least one of G_{i-1} and H_{i-1} is disconnected, then at least one of G_i and H_i is disconnected.

By definition, if z is a vertex of G_i but not of G_{i-1} (i.e. $z \in Z\sim i$), then z is a vertex of $G_{v(W_i)}$. Property 1 of the following lemma says that if (z_1, z_2) is an edge of G_i but not of G_{i-1}, then (z_1, z_2) is an edge of $G_{v(W_i)}$. Property 2 is more technical.

Lemma 6: The following properties hold.

1) If $z \in Z\sim i$, and (z, z') is an edge of G_i, then $z' \in v(W_i)$.

2) If j < i, $z \in Z\sim i$, and Q is a maximal clique of $G_{v(W_i)}$ containing z, then $Q \cap v(W_j) \cap v(W_1) \neq \emptyset$.

Proof:

1) z' is a vertex in G_i, so $z' \in v(W_j)$ for some $j \leq i$. Suppose $z' \notin v(W_i)$. (z, z') is an edge of G, and

therefore $\{z,z'\} \subseteq v(W_k)$ for some $k>i$.

Let Q be a maximal clique of $G_{v(W_k)}$ containing $\{z,z'\}$. $z \in v(W_k) \cap v(W_i)$, and so by Lemma 5 (Property 6), $z \in v(W_k \cup W_i)$. Suppose $W_k \not\subset W_i$. Then $W_i \subset W_i \cup W_k$, and $W_i \cup W_k = W_s$ for some $s<i$. $v(W_s) \subseteq Z_{i-1}$. But $z \in v(W_s)$ and $z \in Z{\sim}i$. Contradiction. Therefore $W_k \subset W_i$. By Lemma 5 (Property 2), $Q \cap v(W_i)$ is a maximal clique of $G_{v(W_i)}$.

$z \in Q \cap v(W_i)$. $z \in Z{\sim}i$, so $z \notin v(W_i \cup W_j)$. Therefore $Q \cap v(W_i)$ is not a maximal clique of $G_{v(W_i \cup W_j)}$, and by Lemma 5 (Property 6), $Q \cap v(W_i) \cap v(W_j)$ is a maximal clique of $G_{v(W_j)}$. But $z' \notin Q \cap v(W_i) \cap v(W_j)$ (because $z' \notin v(W_i)$), and $z' \cup (Q \cap v(W_i) \cap v(W_j))$ is a clique of $G_{v(W_j)}$, contradicting the maximality of $Q \cap v(W_i) \cap v(W_j)$ in $G_{v(W_j)}$.

2) $z \in Z{\sim}i$, and $W_i \subset W_i \cup W_j$, so $z \notin v(W_i \cup W_j)$. Hence Q is not a maximal clique of $G_{v(W_i \cup W_j)}$. By Lemma 5 (Property 4), $Q \cap v(W_j)$ is a maximal clique of $G_{v(W_j)}$. $W_j \subseteq W_1$, so by Lemma 5 (Property 2), $Q \cap v(W_j) \cap v(W_1)$ is a maximal clique of $G_{v(W_1)}$. $v(W_1) \neq \emptyset$, and therefore $Q \cap v(W_j) \cap v(W_1) \neq \emptyset$. □

The following lemma describes two properties of $G_{v(W_i)}$ in the case where G_{i-1} is disconnected, and G_i is connected. These properties, together with the fact that $G_{v(W_i)}$ is P_4-free, will be used later to prove that H_i is disconnected in this case.

Lemma 7: Let $i \in \{2,...,m\}$. If G_{i-1} is disconnected, and G_i is connected, then $G_{v(W_i)}$ is connected, and $G_{v(W_i) \cap Z_{i-1}}$ is disconnected.

Lemma 7 is actually a direct consequence of a stronger result.

Lemma 7′: Let $i \in \{2,...,m\}$. Let C be a connected component of G_i such that C consists of q connected components $\{C_1,C_2,...,C_q\}$ of G_{i-1}, where $q>2$. Then the following are true:

(1) If $C' \in \{C_1,C_2,...,C_q\}$, then $C' \cap v(W_i) \neq \emptyset$.

(2) If $C' \in \{C_1,C_2,...,C_q\}$, then $G_{C' \cap v(W_i)}$ is connected.

(3) $G_{C \cap v(W_i)}$ is connected.

Proof: By Lemma 6 (Property 1), in G_i, the vertices in $Z\sim i$ are only adjacent to vertices in $v(W_i)$. If $C' \in \{ C_1, C_2, ..., C_q \}$, then C' must contain a vertex that is adjacent to a vertex in $Z\sim i$. Therefore, $G_{C' \cap v(W_i)}$ is not empty, and (1) is true.

Claim $7'$.1: If (2) is true for i, then (3) is true for i.

Assume that (2) is true for i. We begin by proving that the vertices in $(C_1 \cup C_2 \cup \cdots C_q) \cap v(W_i)$ are contained in a single connected component of $G_{v(W_i)}$. Let $C' \in \{ C_1, C_2, ..., C_q \}$. Let $c' \in C' \cap v(W_i)$. By (2), there exists a path in $G_{v(W_i)}$ from c' to any other vertex in $v(W_i) \cap C'$.

Let C'' be a member of $\{ C_1, C_2, .., C_q \}$ such that $C'' \neq C'$. Let $c'' \in C'' \cap v(W_i)$. We show there exists a path in $G_{v(W_i)}$ from c' to c''.

Let P be a path in G_i from c' to c''. P can be broken into maximal subpaths $P = P_1, P_2, ..., P_t$ such that each odd numbered subpath consists of vertices in C_s, for some $C_s \in \{ C_1, C_2, ..., C_m \}$, and each even numbered subpath consists of vertices in $Z\sim i$. $Z\sim i \subset v(W_i)$. By Lemma 6 (Property 1), in G_i, vertices in $Z\sim i$ are adjacent only to vertices in $v(W_i)$. Therefore, the last vertex of any odd numbered subpath is contained in $v(W_i)$. Similarly, the first vertex of any odd numbered subpath is contained in $v(W_i)$. If P_{2j+1} is an odd numbered subpath then the first and last vertices of P_{2j+1} are contained in C_s, for some $C_s \in \{ C_1, C_2, ..., C_q \}$. By (2) there exists a path in $G_{v(W_i)}$ joining the first and last vertices of P_{2j+1}. It follows that there exists a path in $G_{v(W_i)}$ joining c' and c''. Hence the vertices in $(C_1 \cup C_2 \cup \cdots C_m) \cap v(W_i)$ are contained in a single connected component of $G_{v(W_i)}$.

We now prove that $G_{C \cap v(W_i)}$ is connected. Let a be any vertex of $C \cap v(W_i)$. Let Q_a be a maximal clique of $G_{v(W_i)}$ containing a. $W_1 = W$, and hence $W_i \subseteq W_1$. By Lemma 5 (Property 2), $Q_a \cap v(W_1)$ is a maximal clique of $G_{v(W_1)} = G_1$. $Q_a \cap v(W_1)$ is contained in $(C_1 \cup C_2 \cup \cdots C_q) \cap v(W_i)$ and hence a is either a member of $(C_1 \cup C_2 \cup \cdots C_q) \cap v(W_i)$ or adjacent to a member of $(C_1 \cup C_2 \cup \cdots C_q) \cap v(W_i)$. Therefore, $G_{C \cap v(W_i)}$ is connected, and Claim $7'$.1 is proved.

The proof of the lemma is by induction on i. The base case is $i=2$. Let $C' \in \{ C_1, C_2, \ldots, C_q \}$. $C' \cap v(W_2) \neq \emptyset$ by an earlier argument. $\{ C_1, C_2, \ldots, C_q \}$ are connected components of $G_1 = G_{v(W_1)}$. $W_2 \subseteq W_1$. By Lemma 5 (Property 8), if $C' \in \{ C_1, C_2, \ldots, C_q \}$, then $G_{C' \cap v(W_2)}$ is connected. Therefore (2) is true for $i=2$. By Claim 7′.1, (3) is also true. Hence Lemma 7′ is true for $i=2$.

Now let $i > 2$. Assume Lemma 7′ is true for all $i' < i$. We will show that (2) is true for i. Lemma 7′ follows.

Let $C' \in \{ C_1, C_2, \ldots, C_q \}$. By Lemma 5 (Property 1), every vertex in G is either in Z_1, or adjacent to a vertex in Z_1. Therefore, every connected component of G_{i-1} must contain at least one connected component of G_1. Assume first that C' contains only one connected component, E, of G_1. By Lemma 5 (Property 8), $G_{E \cap v(W_i)}$ is connected or empty. Let a be any vertex in $C' \cap v(W_i)$. Let Q_a be a maximal clique of $G_{v(W_i)}$ containing a. By Lemma 5 (Property 2), $Q_a \cap v(W_1)$ is a maximal clique of G_1. $Q_a \cap v(W_1) \subseteq E \cap v(W_i)$. Thus $E \cap v(W_i) \neq \emptyset$, so $G_{E \cap v(W_i)}$ is connected. Furthermore, because $a \in Q_a$, a is either a member of $E \cap v(W_i)$, or adjacent to a member of $E \cap v(W_i)$. Therefore, $G_{C' \cap v(W_i)}$ is connected.

Assume now that C' contains more than one connected component of G_1. Let $E_1, E_2, \ldots, E_t$ be the connected components of G_1 that are contained in C'. $E_1, E_2, \ldots, E_t$ are in a single connected component of G_{i-1}, but are distinct components of G_1. Therefore, there exists a $j < i$ such that $E_1, E_2, \ldots, E_t$ are not contained in a single connected component of G_{j-1}, but they are contained in a single connected component $D \subseteq C'$ of G_j. Let $\{ D_1, D_2, \ldots, D_u \}$ be the connected components of G_{j-1} that are contained in D.

Suppose $W_i \subset W_j$. By the inductive hypothesis, $G_{D \cap v(W_j)}$ is connected. Therefore by Lemma 5 (Property 8), $G_{D \cap v(W_j) \cap v(W_i)}$ is either empty or connected. Let a be any vertex in $C' \cap v(W_i)$. Let Q_a be a maximal clique of $G_{v(W_i)}$ containing a. By Lemma 5 (Property 2), $Q_a \cap v(W)$ is a maximal clique of $G_{v(W)}$. $W_i \subset W_j \subset W$, and $Q_a \cap v(W) \subseteq v(W_i) \cap v(W)$, so by Lemma 5 (Property 5), $Q_a \cap v(W) \subseteq v(W_j)$. $Q_a \cap v(W) \subseteq D \cap v(W_i)$. Hence, $Q_a \cap v(W) \subseteq D \cap v(W_j) \cap v(W_i)$. It follows that $D \cap v(W_j) \cap v(W_i) \neq \emptyset$ and $G_{D \cap v(W_j) \cap v(W_i)}$ is connected. Furthermore, because $a \in Q_a$, a is either a member of $D \cap v(W_j) \cap v(W_i)$, or adjacent to a member of $D \cap v(W_j) \cap v(W_i)$. Therefore $G_{C' \cap v(W_i)}$ is connected.

Suppose, on the other hand, that $W_i \not\subset W_j$. We will show that this assumption leads to a contradiction.

As we mentioned before, in G_i, vertices in $Z_{\sim i}$ are adjacent only to vertices in $v(W_i)$. It follows that if $C' \in \{ C_1, C_2, \ldots, C_q \}$, then there exists $C'' \in \{ C_1, C_2, \ldots, C_q \}$, $c' \in C' \cap v(W_i)$, and $c'' \in C'' \cap v(W_i)$, such that $C' \neq C''$, and there exists a path in $G_{v(W_i)}$ joining c' and c''. Let $P = c' z_1 z_2 \cdots z_r c''$ be such a path.

c' and c'' are connected by a path in $G_{v(W_i)}$, $G_{v(W_i)}$ is P_4-free, and c' and c'' are contained in distinct connected components of G_{i-1}. It follows that there exists a $z' \in Z_{\sim i}$ such that $c'' z' c'$ is a path in $G_{v(W_i)}$.

Let $Q_{z'}$ be a maximal clique of $G_{v(W_i)}$ containing $\{ z', c' \}$. By Lemma 6 (Property 2), $Q_{z'} \cap v(W_j) \cap v(W) \neq \emptyset$. Furthermore $Q_{z'} \cap v(W_j) \cap v(W) \subseteq D \cap v(W_j) \cap v(W_i)$. Let $e' \in Q_{z'} \cap v(W_j) \cap v(W)$. $z' \in Z_{\sim i}$, so $z' \neq e'$. $z' \in Q_{z'}$, so z' is adjacent to e'.

Let Γ_1 be the connected component of $G_{C' \cap v(W_i)}$ containing e'. Let Γ_2 be the connected component of $G_{C'' \cap v(W_i)}$ containing c''. There are no edges joining vertices of Γ_1 and Γ_2. $z' \in Z_{\sim i}$, so $z' \notin \Gamma_1 \cup \Gamma_2$. It follows from Lemma 5 (Property 7) that z' is adjacent to all vertices in Γ_1.

Remember that, by the induction hypothesis, $G_{D \cap v(W_j)}$ is connected. Let $D' \in \{ D_1, D_2, \ldots, D_u \}$ be the connected component of G_{j-1} containing e'. Let $d'' \in D'' \cap v(W_j)$, where $D'' \in \{ D_1, D_2, \ldots, D_u \}$, and $D' \neq D''$. $G_{v(W_j)}$ is P_4-free, and e' and d'' appear in distinct connected components of G_{j-1}. It follows that there exists a $z'' \in Z_{\sim j}$ such that $e' z'' d''$ is a path in $G_{v(W_j)}$. $c'' z' e' z''$ is a path in G_i.

Let $Q_{z''}$ be a maximal clique of $G_{v(W_j)}$ containing $\{ e', z'' \}$. Because $W_i \not\subset W_j$, $W_j \subset W_i \cup W_j$, which implies that $W_i \cup W_j = W_k$ for some $k < j$. $z'' \in Z_{\sim j}$, and hence $z'' \notin v(W_i \cup W_j)$. Therefore, $Q_{z''}$ is not a maximal clique of $G_{v(W_i \cup W_j)}$, and by Lemma 5 (Property 4), $Q_{z''} \cap v(W_i)$ is a maximal clique of $G_{v(W_i)}$. But $Q_{z''} \cap v(W_i) \subseteq \Gamma_1$, and so $\{ z' \} \cup (Q_{z''} \cap v(W_i))$ is a clique of $G_{v(W_i)}$. This contradicts the maximality of $Q_{z''} \cap v(W_i)$. □

In contrast to the situation of G_{i-1} and G_i, if (z_1, z_2) is an edge of H_i but not of H_{i-1}, then (z_1, z_2) is not necessarily an edge in $H_{v(W_i)}$. However, the following lemma allows us to ignore this difficulty.

Lemma 8: If P is a path in H_i from z_1 to z_2, then there exists a path P' in H_i from z_1 to z_2 such that every edge in P' is either in H_{i-1} or $H_{v(W_i)}$.

Proof: We show that for every edge (a, b) in H_i such that $a \in Z\sim i$ and $b \notin v(W_i)$, there exists a path $P' = az''z'b$ such that (a, z'') is an edge in $H_{v(W_i)}$, and (z'', z') and (z', b) are edges in H_{i-1}. The lemma follows immediately.

Let a and b be as described above. Let j be such that $b \in Z\sim j$. We show first that $W_i \not\subset W_j$. (a, b) is an edge in H_i, yet (a, b) is not an edge in $H_{v(W_i)}$. Therefore, there exists a $k > i$ such that (a, b) is an edge in $H_{v(W_k)}$. $a \in v(W_k) \cap v(W_i)$. By Lemma 5 (Property 6), $v(W_k) \cap v(W_i) \subseteq v(W_k \cup W_i)$. Suppose $W_k \not\subset W_i$. Then $W_k \subset W_i \cup W_k$, and $W_k \cup W_i = W_s$ for some $s < i$. $v(W_s) \subseteq Z_{i-1}$. But $a \in v(W_s)$ and $a \in Z\sim i$. Contradiction. Therefore, $W_k \subset W_i$. If $W_i \subset W_j$, then $W_k \subset W_i \subset W_j$ and $b \in (v(W_k) \setminus v(W_i)) \cap v(W_j)$, contradicting Lemma 5 (Property 5). Therefore, $W_i \not\subset W_j$.

It follows that $W_i \subset W_i \cup W_j$ and $W_j \subset W_i \cup W_j$. We need the following claim.

Claim 8.1: Let G' be a P_4-free graph on the vertex set Z'. Let A and B be subsets of Z' such that A and B both contain at least one maximal clique of G', and if Q is a maximal clique of G', then Q is not contained in both A and B. Then there exist $z' \in A \setminus B$, and $z'' \in B \setminus A$ such that z' and z'' are not adjacent in G'.

Pf of claim: The proof is by induction on $|Z'|$. Clearly $|Z'| > 1$. If $Z' = \{z', z''\}$, then the conditions of the claim are only satisfied if $A = \{z'\}$, $B = \{z''\}$, and z' and z'' are not adjacent. So, the claim is true when $|Z'| = 2$.

Let $i > 2$. Assume the claim is true when $|Z'| < i$. Show that it is true when $|Z'| = i$. Because G' is

P_4-free, by Lemma 1, either G' is disconnected, or $\overline{G'}$, the complement of G', is disconnected.

Suppose G' is disconnected. Let Z'_1, Z'_2 be a non-trivial partition of Z' such that G' contains no edges between vertices in Z'_1 and vertices in Z'_2. Define $G'_{Z'_1}$ and $G'_{Z'_2}$ to be the subgraphs of G' induced by Z'_1 and Z'_2 respectively. A and B do not contain a common maximal clique of G', and therefore A and B do not contain a common maximal clique of either $G'_{Z'_1}$ or $G'_{Z'_2}$. Every maximal clique of G' is contained either in Z'_1 or in Z'_2. Suppose A contains a maximal clique Q_1 of $G'_{Z'_1}$, and B contains a maximal clique Q_2 of $G'_{Z'_2}$ (or vice versa). $Q_1 \not\subset B$ and $Q_2 \not\subset A$. Let $z' \in Q_1 \backslash B$, $z'' \in Q_2 \backslash A$. z' and z'' are not adjacent, and the claim is proved for this case. If, on the other hand, A and B do not contain cliques in opposite sides of the partition Z'_1, Z'_2, then the maximal cliques contained in A and B are all contained in one side of the partition, say Z'_1. The claim is then true by the inductive hypothesis applied to $G'_{Z'_1}$, Z'_1, $A \cap Z'_1$, and $B \cap Z'_1$.

Suppose now that $\overline{G'}$ is disconnected. Let Z'_1,Z'_2 be a non-trivial partition of Z' such that $\overline{G'}$ contains no edges between vertices in Z'_1 and vertices in Z'_2. In G' every vertex in Z'_1 is adjacent to every vertex in Z'_2. Suppose A and B contain a common maximal clique Q' of $G'_{Z'_1}$, and a common maximal clique Q'' of $G'_{Z'_2}$. $Q' \cup Q''$ is a maximal clique of G'. But A and B do not contain a common maximal clique of G'. Contradiction. Therefore, either A and B do not contain a common maximal clique of $G'_{Z'_1}$, or A and B do not contain a common maximal clique of $G'_{Z'_2}$. Without loss of generality, assume the former is true. Since every maximal clique of G is of the form $Q' \cup Q''$, where Q' is a maximal clique of $G'_{Z'_1}$, and Q'' is a maximal clique of $G'_{Z'_2}$, A and B contain maximal cliques in both $G'_{Z'_1}$ and $G'_{Z'_2}$. The claim is then true by the inductive hypothesis applied to $G'_{Z'_1}$,Z',$A \cap Z'_1$, and $B \cap Z'_1$.

We can now prove the existence of the path $az''z'b$ in H_i, a maxterm graph. To do this, we switch our attention for the moment to some minterm graphs. $W_i \subset W_i \cup W_j$ and $W_j \subset W_i \cup W_j$, $a \in Z{\sim}i$ and $b \in Z{\sim}j$, so a and b are not members of $\nu(W_i \cup W_j)$. By Lemma 5 (Property 6), $\nu(W_i) \cap \nu(W_j) \subseteq \nu(W_i \cup W_j)$, and therefore a and b are not members of $\nu(W_i) \cap \nu(W_j)$. Let A be the set of vertices of $\nu(W_i) \cap \nu(W_j)$ that are adjacent to a in $G_{\nu(W_i)}$. Let B be the set of vertices of $\nu(W_i) \cap \nu(W_j)$ that are adjacent to b in $G_{\nu(W_j)}$.

If Q_a is a maximal clique of $G_{v(W_i)}$ containing a, it follows from Lemma 5 (Property 4) that $Q_a \cap v(W_j)$ is a maximal clique of $G_{v(W_j)}$. $Q_a \cap v(W_j) \subseteq v(W_i) \cap v(W_j)$, and so $Q_a \cap v(W_j)$ is also a maximal clique of $G_{v(W_i)\cap v(W_j)}$. $Q_a \cap v(W_j) \subseteq A$, so A contains at least one maximal clique of $G_{v(W_i)\cap v(W_j)}$. Similarly, B contains at least one maximal clique of $G_{v(W_i)\cap v(W_j)}$.

If $Q' \subseteq A$ is a maximal clique of $G_{v(W_i)\cap v(W_j)}$, then there exists a maximal clique $Q'_a \supset Q'$ of $G_{v(W_i)}$ such that $a \in Q'_a$. If $Q' \subseteq B$ is a maximal clique of $G_{v(W_i)\cap v(W_j)}$, then there exists a maximal clique $Q'_b \supset Q'$ of $G_{v(W_j)}$ such that $b \in Q'_b$.

It follows that every maximal clique of $G_{v(W_i)\cap v(W_j)}$ contained in A is a maximal clique of $G_{v(W_j)}$ and is therefore not contained in B. Similarly every maximal clique of $G_{v(W_i)\cap v(W_j)}$ contained in B is a maximal clique of $G_{v(W_i)}$ and is therefore not contained in A. $G_{v(W_i)}$ is P_4-free, and hence $G_{v(W_i)\cap v(W_j)}$ is P_4-free. $G_{v(W_i)\cap v(W_j)}$, A, and B, satisfy the conditions of Claim 8.1, and therefore there exist $z' \in A \setminus B$, $z'' \in B \setminus A$ such that z' and z'' are not adjacent in $G_{v(W_i)\cap v(W_j)}$. a, z', and z'' are vertices of $G_{v(W_i)}$, a and z'' are not adjacent in $G_{v(W_i)}$, and z' and z'' are not adjacent in $G_{v(W_i)}$. $H_{v(W_i)}$ is the complement of $G_{v(W_i)}$, and hence (a, z''), and (z', z'') are edges in $H_{v(W_i)}$. z' and b are vertices of $G_{v(W_j)}$, they are not adjacent in $G_{v(W_j)}$, and hence they are adjacent in $H_{v(W_j)}$. Therefore, a, z'', z', b is a path in H_i. □

We are now ready to prove Lemmas 9 and 10, which together compose the inductive step of our proof that at least one of G and H are disconnected.

Lemma 9: If $i \in \{2,...,m\}$, and H_{i-1} is disconnected, then H_i is disconnected.

Proof: Assume not. By Lemma 8, if two vertices of H_i are joined by a path in H_i, then they are joined by a path all of whose edges are contained in H_{i-1} or $H_{v(W_i)}$. Let $P = az_1z_2 \cdots z_q b$ be the shortest path in H_i that joins two connected components of H_{i-1}, and has the property that all of its edges are in H_{i-1} or $H_{v(W_i)}$. Clearly $z_1, z_2, ..., z_q$ are vertices in $Z \sim i$. Therefore, a and b are vertices in $v(W_i)$, and all edges in P are contained in $G_{v(W_i)}$. $G_{v(W_i)}$ is P_4-free, a and b are not adjacent, and P is as short as possible, so $P = az_1b$.

Let C_1 and C_2 be the connected components of H_{i-1} containing a and b respectively. Let A be the connected component of $H_{C_1 \cap v(W_i)}$ containing a. Let B be the connected component of $H_{C_2 \cap v(W_i)}$ containing b. There are no edges between vertices of A and vertices of B. $z_1 \notin Z_{i-1}$, so $z_1 \notin A$ and $z_1 \notin B$. Therefore, by Lemma 5 (Property 7), z_1 is adjacent to all vertices of A and B.

Let j be such that $a \in Z\sim_j$. $j < i$. $a \in v(W_j) \cap v(W_i)$, and therefore $a \in v(W_j \cup W_i)$. Because $a \in Z\sim_j$, $W_j \cup W_i = W_j$, and therefore $W_i \subset W_j$. Let Q_a be a maximal clique of $H_{v(W_j)}$ containing a. By Lemma 5 (Property 3), $Q_a \cap v(W_i)$ is a maximal clique of $H_{v(W_i)}$. But $Q_a \cap v(W_i) \subseteq A$, z_1 is adjacent to all vertices in A, $z_1 \notin A$, and $\{ z_1 \} \cup (Q_a \cap v(W_i))$ is a clique in $H_{v(W_i)}$. Contradiction. Therefore, if H_{i-1} is disconnected, then H_i is disconnected. □

Lemma 10: If $i \in \{ 2,...,m \}$, G_{i-1} is disconnected, and G_i is connected, then H_i is disconnected.

Proof: By Lemma 7, $G_{v(W_i)}$ is connected, and $G_{v(W_i) \cap Z_{i-1}}$ is disconnected.

By Lemma 4, $G_{v(W_i)}$ is P_4-free and connected, and $H_{v(W_i)}$ is the complement of $G_{v(W_i)}$. The complement of a disconnected graph is a connected graph, so $H_{v(W_i) \cap Z_{i-1}}$ is connected. Lemma 1 says that the complement of a connected P_4-free graph with more than one vertex is a disconnected graph, so $H_{v(W_i)}$ is disconnected. Therefore, $H_{v(W_i)}$ contains at least two connected components, one of which is a connected component C which consists wholly of vertices in $Z\sim_i$.

Let $a \in C$. Clearly, in H_i, a is not adjacent to any vertices in $v(W_i) - C$. Suppose, in H_i, a is adjacent to some vertex b such that $b \notin v(W_i)$. By Lemma 8, there exists a path $P = a z_1 z_2 \cdots z_q b$ in H_i from a to b such that P consists of edges in H_{i-1} and $H_{v(W_i)}$. Let (z', z'') be the first edge in P with the property that $z' \in Z\sim_i$, and $z'' \notin Z\sim_i$. (z', z'') must be an edge in $H_{v(W_i)}$, so $z'' \in v(W_i) - Z\sim_i$. But then C contains a vertex in $v(W_i) - Z\sim_i$. Contradiction. □

The main lemma now follows easily.

Lemma 11: For all $i \in \{ 1,...,m \}$, at least one of G_i and H_i is disconnected.

Proof: By induction on i. $G_1 = G_{V(W)}$, $H_1 = H_{V(W)}$, and by Lemmas 1 and 4 at least one of G_1 and H_1 is disconnected. Let $i > 1$. Assume the lemma is true for $i' < i$. Show it is true for i.

By the inductive assumption, at least one of G_{i-1} and H_{i-1} is disconnected. If H_{i-1} is disconnected, then by Lemma 9, H_i is disconnected. Suppose G_{i-1} is disconnected. If G_i is also disconnected, then there is nothing to prove. If G_i is connected, then by Lemma 10, H_i is disconnected. □

5. Conclusions

Theorem 3 now follows immediately from Theorem 2, and the lemmas. For the benefit of the reader, we review the structure of the proof.

Theorem 3: Let $f : \{0,1\}^V \rightarrow \{0,1\}$ be a monotone boolean function, and let $Z \subseteq V$, such that for all $S \in MIN(f)$, $|S \cap (V-Z)| \leq 1$. Then f is read-once on Z iff for all $S \in MIN(f)$, $T \in MAX(f)$, $|S \cap T \cap Z| \leq 1$.

Proof: ☞ By Theorem 2, if f is read-once on Z, then for all $Z' \subseteq Z$ such that $|Z'| > 1$, either the subgraph of the minterm graph of f induced by Z' is disconnected, or the subgraph of the maxterm graph of f induced by Z' is disconnected. This direction of the proof follows by applying this fact to all $Z' \subseteq Z$ such that $|Z'| = 2$.

☜ By Lemma 11, either the subgraph of the minterm graph of f induced by Z is disconnected, or the subgraph of the maxterm graph of f induced by Z is disconnected. By Lemma 3, this fact is sufficient to prove the theorem. □

In this paper, we have shown two conditions for f to be read-once on Z. Condition 2 (on the connectedness of subgraphs of the minterm and maxterm graphs) is necessary and sufficient. Condition 1 ($V-S \in MIN(f), V-T \in MAX(f), |S \cap T \cap Z| \leq 1$) is necessary but not always sufficient. In Theorem 3, we have shown a class of functions for which Condition 1 is sufficient for f to be read-once on Z. There are functions which are not in this class but for which Condition 1 is nonetheless sufficient. Consider, for example, $f = xy + wx + zy$ and $Z = \{w, z\}$. An open question is to characterize those f and Z

for which Condition 1 is sufficient for f to be read-once on Z.

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