Fourier Integrals and the Discrete Fourier Transform

Our exposition follows that in "Numerical Recipes in C - The Art of Scientific Computing" by W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, 2nd edition, Cambridge University press, 1992.

Fourier Integrals

Let $h(t)$ be a time-dependent signal. The following weighted integral over time t is the Fourier transform of $h(t)$:

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} dt,$$

with frequency $f \in (-\infty \ \infty)$. The inverse Fourier transform is

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{2\pi i f t} df,$$

Alternate definition has the sign of i reversed in the above expressions. This choice of sign affect the definition of the discrete Fourier transform which will be introduced later. Our sign convention here agrees with that in Heath's book, and with Matlab.

If the Fourier transform of $g(t)$ is $G(f)$, then the convolution of g and h is defined to be the following function of t:

$$g * h = \int_{-\infty}^{\infty} g(\tau)\ h(t-\tau) d\tau = \int_{-\infty}^{\infty} h(\tau)\ g(t-\tau) = h * g.$$

The total power carried by the signal is

$$P = \int_{-\infty}^{\infty} |h(t)|^2 dt = \int_{-\infty}^{\infty} |H(f)|^2 df.$$

Fourier Transform of Discretely Sampled Data

In reality we often cannot record the signal $h(t)$ continuously. We can only sample it discretely. Very often the signal is sampled at evenly spaced intervals of time Δ. The sequence of sampled values is given by:

$$h_n = h(n\Delta), \qquad n = \cdots, -2, -1, 0, 1, 2, \cdots.$$

The sample rate, defined as the number of samples recorded per unit time, is given by Δ^{-1}. The Nyquist critical frequency is defined by

$$f_c = \frac{1}{2\Delta}.$$

The Nyquist critical frequency plays a very crucial role in discrete Fourier transform because of the sampling theorem and the effects of aliasing, as we will discuss next.

Sampling Theorem

If a continuous function $h(t)$ sampled at time interval Δ happens to be bandwidth limited to frequencies bounded by f_c in magnitude, *i.e.* $H(f) = 0$ for $|f| \geq f_c$, then the function $h(t)$ is completely determined by the sampled points, h_n. The entire continuous function $h(t)$ can be constructed from h_n explicitly by the following formula:

$$h(t) = \Delta \sum_{n=-\infty}^{\infty} h_n \frac{\sin\left[2\pi f_c(t - n\Delta)\right]}{\pi(t - n\Delta)}.$$

Thus the entire information content of a bandwidth limited signal can be recorded by sampling it at a rate $\Delta^{-1} = 2f_c$.

To an extend, most signals are bandwidth limited. Passing a signal through an amplifier often causes it to be bandwidth limited since amplifiers often cannot respond to the signal when the frequency is sufficiently high (the electrons in the device simply cannot move that fast).

Effects of Eliasing

The very act of discretely sampling a continuous function that is not bandwidth limited to less than f_c spuriously moves (falsely translates) the spectral density of any frequency components outside $[-f_c\ f_c]$ into that range. This phenomenon is called aliasing.

Note that 2 separate signals $e^{2\pi i f_1 t}$ and $e^{2\pi i f_2 t}$ whose frequencies differ by a multiple of Δ^{-1} give exactly the same samples if data are recorded at time interval Δ. To see that, let us assume that $f_2 = f_1 + k/\Delta$ and $t = n\Delta$. Thus

$$e^{-2\pi i f_2 t} = e^{-2\pi i(f_1+k/\Delta)t} = e^{-2\pi i(f_1+k/\Delta)n\Delta} = e^{-2\pi i f_1 n\Delta} = e^{-2\pi i f_1 t}.$$

Discrete Fourier Transform

Suppose we have N consecutively sampled values

$$h_k = h(t_k) = h(k\Delta), \qquad k = 0, 1, \cdots, N-1,$$

thus the sampling interval is Δ. To simplify out discussion, we assume that N is even. We want to estimate the Fourier transform at N discrete values in $[-f_c\ f_c]$. The frequency points are then given by

$$f_m = \frac{m}{N\Delta}, \qquad m = -\frac{N}{2}, \cdots, -1, 0, 1, \cdots, \frac{N}{2}.$$

We approximate the Fourier integral by a Riemann sum:

$$H(f_m) = \int_{-\infty}^{\infty} h(t)\ e^{-2\pi i f_m t} dt = \sum_{k=0}^{N-1} h(t_k) e^{-2\pi i f_m t_k} \Delta = \Delta \sum_{k=0}^{N-1} h_k e^{-2\pi i \frac{m}{N\Delta} k\Delta} = \Delta H_m,$$

where

$$H_m = \sum_{k=0}^{N-1} h_k e^{-2\pi i m k/N}$$

is the discrete Fourier transform of the sampled data.

Note that H_m is periodic in m with period N because

$$H_{m+N} = \sum_{k=0}^{N-1} h_k e^{-2\pi i(m+N)k/N} = \sum_{k=0}^{N-1} h_k e^{-2\pi imk/N} = H_m.$$

Therefore we see that

$$H_{-\frac{N}{2}} = H_{-\frac{N}{2}+N} = H_{\frac{N}{2}},$$

and so there are exactly N independent Fourier components (barring any symmetries that the original signal may have).

The inverse Fourier transform is given by

$$h_k = \frac{1}{N}\sum_{m=0}^{N-1} H_m e^{2\pi imk/N}.$$

Using exactly the argument, we can easily see that h_k must also be period in k with period N. This is true even if the original signal may not be periodic in the first place.

It is customary to shift the range for m from

$$m = -\frac{N}{2}, \cdots, -1, 0, 1, \cdots, \frac{N}{2} - 1, \frac{N}{2}$$

to

$$m = 0, 1, 2, \cdots, N-1,$$

to agree with the usual definition of the discrete Fourier transform and its inverse transform. This is done by taking the first $N/2$ element of the original sequence $m = -\frac{N}{2}, \cdots, -1$ and adding N to each element to obtain $m = \frac{N}{2}, \cdots, N-1$, which is then placed at the end of the original sequence. The Nyquist frequency then corresponds to $m = N/2$.

The discrete version of Parseval's theorem is

$$\sum_{k=0}^{N-1} |h_n|^2 = \frac{1}{N}\sum_{m=0}^{N-1} |H_m|^2.$$