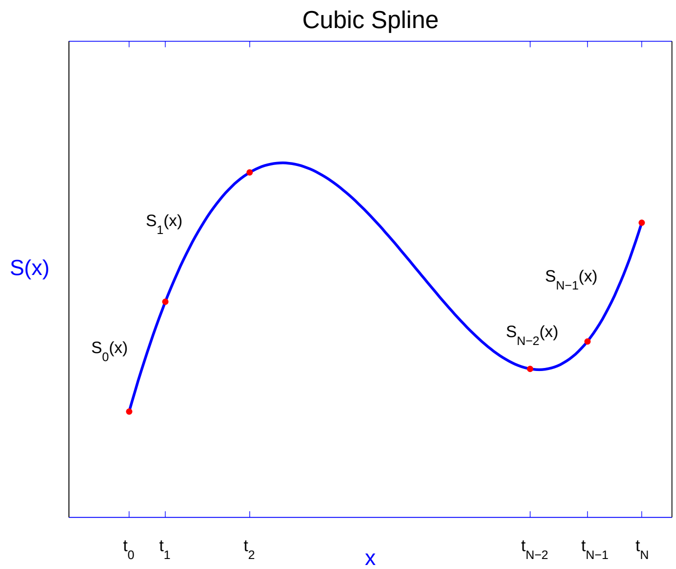
Interpolation using Cubic Spline

Given $N+1$ data points in the interval $[a, b]$,

x	t_0	t_1	$\cdots$	t_N
y	y_0	y_1	$\cdots$	y_N



we want to construct a cubic spline $S(x)$ to interpolate the table of values presumably of a function $f(x)$. We assume that the points are ordered so that

$$a = t_0 < t_1 < \cdots < t_N = b.$$

$S(x)$ is given by a different cubic polynomial in each interval $[t_0, t_1]$, $[t_1, t_2]$, $\cdots$, $[t_{N-1}, t_N]$. Let $S(x)$ be given by $S_i(x)$ if $x \in [t_i, t_{i+1}]$. Each cubic polynomial is defined by 4 coefficients and so we have a total of $4N$ parameters. These are determined by the following conditions:

1. $S(x)$ must interpolate the data points and so in each subinterval $i = 0, \cdots, N-1$, we must have $S_i(t_i) = y_i$ and $S_i(t_{i+1}) = y_{i+1}$.
2. $S'(x)$ must be continuous at each of the internal knots. Therefore for $i = 1, 2, \cdots, N-1$ we must have $S'_{i-1}(t_i) = S'_i(t_i)$.
3. $S''(x)$ must be continuous at each of the internal knots. Therefore for $i = 1, 2, \cdots, N-1$ we must have $S''_{i-1}(t_i) = S''_i(t_i)$.
4. A choice of one of the following 2 conditions at the 2 end points a and b:
   (a) The natural spline: $S''_0(a) = 0 = S''_{N-1}(b)$,
   (b) The clamped cubic spline: $S'_0(a) = f'(a)$ and $S'_{N-1}(b) = f'(b)$.

The clamped cubic spline gives more accurate approximation to the function $f(x)$, but requires knowledge of the derivative at the endpoints. Condition 1 gives $2N$ relations. Conditions 2 and 3 each gives $N-1$ relations. Together with the 2 relations from condition 4, we have a total of $2N + 2(N-1) + 2 = 4N$ conditions. Thus we have just the right number of relations to determined all the parameters uniquely.

The best way to express the cubic polynomial within each subinterval is to note that since $S_i(x)$ is a cubic polynomial, then $S_i''(x)$ must be a linear function of the form

$$S_i''(x) = \alpha x + \beta.$$

Next we denote $S_i''(t_i) = z_i$ and $S_i''(t_{i+1}) = z_{i+1}$, so that replacing i by $i-1$ in the second expression gives $S_{i-1}''(t_i) = z_i$, and therefore condition 3 is automatically satisfied. We then evaluate $S_i''(x)$ at the endpoint of the subinterval to get

$$S_i''(t_i) = z_i = \alpha t_i + \beta$$

$$S_i''(t_{i+1}) = z_{i+1} = \alpha t_{i+1} + \beta.$$

We can then solve for α and β to obtain

$$\alpha = \frac{z_{i+1} - z_i}{h_i} \qquad \beta = \frac{z_i t_{i+1} - z_{i+1} t_i}{h_i},$$

where we define $h_i = t_{i+1} - t_i$ for $i = 0, \cdots, N-1$. Inserting these back to the expression for $S_i''(x)$ gives

$$S_i''(x) = \frac{z_i}{h_i}(t_{i+1} - x) + \frac{z_{i+1}}{h_i}(x - t_i).$$

Integrating this expression, we have

$$S_i'(x) = -\frac{z_i}{2h_i}(t_{i+1} - x)^2 + \frac{z_{i+1}}{2h_i}(x - t_i)^2 + p,$$

where p is a constant. Integrating one more time, we have

$$S_i(x) = -\frac{z_i}{6h_i}(t_{i+1} - x)^3 + \frac{z_{i+1}}{6h_i}(x - t_i)^3 + px + q,$$

where q is another constant. Instead of using constants p and q, it is better to use constants C and D so that

$$S_i(x) = -\frac{z_i}{6h_i}(t_{i+1} - x)^3 + \frac{z_{i+1}}{6h_i}(x - t_i)^3 + C(x - t_i) + D(t_{i+1} - x).$$

Evaluating this at t_i gives

$$S_i(t_i) = y_i = \frac{z_i}{6}h_i^2 + Dh_i,$$

from which we get $D = \frac{y_i}{h_i} - \frac{z_i h_i}{6}$. Similarly, evaluating $S_i(x)$ at t_{i+1} gives

$$S_i(t_{i+1}) = y_{i+1} = \frac{z_{i+1}}{6}h_i^2 + Ch_i,$$

from which we get $C = \frac{y_{i+1}}{h_i} - \frac{z_{i+1} h_i}{6}$. Using these results for C and D, we finally have the expression for $S_i(x)$:

$$S_i(x) = -\frac{z_i}{6h_i}(t_{i+1}-x)^3 + \frac{z_{i+1}}{6h_i}(x-t_i)^3 + \left(\frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6}\right)(x-t_i) + \left(\frac{y_i}{h_i} - \frac{z_i h_i}{6}\right)(t_{i+1}-x). \quad (1)$$

Next we have to impose condition 2 on $S_i'(x)$ which can be obtained from the above equation by differentiation:

$$S_i'(x) = -\frac{z_i}{3h_i}(t_{i+1}-x)^2 + \frac{z_{i+1}}{2h_i}(x-t_i)^2 + \frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6} + \frac{y_i}{h_i} - \frac{z_i h_i}{6}. \tag{2}$$

We are going to apply the condition $S_i'(t_i) = S_{i-1}'(t_i)$, for $i = 1, 2, \cdots, N-1$ (at all internal knots). First at $t = t_i$ we have

$$\begin{aligned} S_i'(t_i) &= -\frac{z_i h_i}{3} + \frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6} - \frac{y_i}{h_i} + \frac{z_i h_i}{6} \\ &= -\frac{z_i h_i}{3} + \frac{y_{i+1}}{h_i} - \frac{z_{i+1}h_i}{6} - \frac{y_i}{h_i} \end{aligned}$$

By replacing the index i by $i-1$ in Eq. (2), we have

$$S_{i-1}'(x) = -\frac{z_{i-1}}{3h_{i-1}}(t_i-x)^2 + \frac{z_i}{2h_{i-1}}(x-t_{i-1})^2 + \frac{y_i}{h_{i-1}} - \frac{z_i h_{i-1}}{6} + \frac{y_{i-1}}{h_{i-1}} - \frac{z_{i-1}h_{i-1}}{6}. \tag{3}$$

Substituting $t = t_i$ yields

$$\begin{aligned} S_{i-1}'(t_i) &= -\frac{z_i h_{i-1}}{3} + \frac{y_i}{h_{i-1}} - \frac{z_i h_{i-1}}{6} - \frac{y_{i-1}}{h_{i-1}} + \frac{z_{i-1}h_{i-1}}{6} \\ &= -\frac{z_i h_{i-1}}{3} + \frac{y_i}{h_{i-1}} - \frac{y_{i-1}}{h_{i-1}} + \frac{z_{i-1}h_{i-1}}{6} \end{aligned}$$

The condition $S_i'(t_i) = S_{i-1}'(t_i)$ then gives the result

$$h_{i-1}z_{i-1} + u_i z_i + h_i z_{i+1} = v_i, \tag{4}$$

where

$$u_i = 2(h_i + h_{i-1}) \quad b_i = 6\frac{(y_{i+1}-y_i)}{h_i} \quad v_i = b_i - b_{i-1}, \tag{5}$$

and $i = 1, 2, \cdots, N-1$.

In the case of the natural cubic spline, we set $z_0 = 0 = z_N$ to obtain the following linear system of equations:

$$\begin{bmatrix} u_1 & h_1 & & & & \\ h_1 & u_2 & h_2 & & & \\ & h_2 & u_3 & h_3 & & \\ & & \cdot & \cdot & \cdot & \\ & & & h_{N-3} & u_{N-2} & h_{N-2} \\ & & & & h_{N-2} & u_{N-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \cdot \\ z_{N-2} \\ z_{N-1} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \cdot \\ v_{N-2} \\ v_{N-1} \end{bmatrix}. \tag{6}$$

It can be solved for the vector $(z_1, z_2, \cdot, z_{N-1})$ using Gaussian elimination without pivoting.

After determining the coefficients $(z_0, z_1, \cdot, z_N)$, any value of the cubic spline can then be computed from Eq. (1). Given any x, it is necessary first to find which of the intervals

$$(-\infty, t_1), [t_1, t_2), \cdots, [t_{N-2}, t_{N-1}), [t_{N-1}, \infty)$$

contains x. This determines the $S_i(x)$ to be used for that interval. $S_i(x)$ can be computed efficiently using Horner's nested procedure. We need to rewrite $S_i(x)$ in a nested form:

$$S_i(x) = y_i + (x-t_i)\left[C_i + (x-t_i)[B_i + (x-t_i)A_i]\right].$$

One can easily see that

$$A_i = \frac{z_{i+1} - z_i}{6h_i} \qquad B_i = \frac{z_i}{2} \qquad C_i = -\frac{h_i z_{i+1}}{6} - \frac{h_i z_i}{3} + \frac{y_{i+1} - y_i}{h_i}.$$