ASSIGNMENT 3 - Solution

Problem 4

A straightforward iteration of the sequence defined recursively by

$$x_{n+1} = 2^{n+1}\left[\sqrt{1+2^{-n}x_n} - 1\right]$$

starting from $x_0 = 2$ yields the following result

```
n =  1    x(n) = 1.4641016151377544    error =  3.65e-001
n =  2    x(n) = 1.2642960518099695    error =  1.66e-001
n =  3    x(n) = 1.1776215235190168    error =  7.90e-002
n =  4    x(n) = 1.1372077291666329    error =  3.86e-002
n =  5    x(n) = 1.1176885465556694    error =  1.91e-002
n =  6    x(n) = 1.1080957635770972    error =  9.48e-003
n =  7    x(n) = 1.1033404504923681    error =  4.73e-003
n =  8    x(n) = 1.1009729865915574    error =  2.36e-003
n =  9    x(n) = 1.0997917932434120    error =  1.18e-003
n = 10    x(n) = 1.0992018300289601    error =  5.90e-004
n = 11    x(n) = 1.0989070066380009    error =  2.95e-004
n = 12    x(n) = 1.0987596344784833    error =  1.47e-004
n = 13    x(n) = 1.0986859582808393    error =  7.37e-005
n = 14    x(n) = 1.0986491226503858    error =  3.68e-005
n = 15    x(n) = 1.0986307054554345    error =  1.84e-005
n = 16    x(n) = 1.0986214970180299    error =  9.21e-006
n = 17    x(n) = 1.0986168928211555    error =  4.60e-006
n = 18    x(n) = 1.0986145907663740    error =  2.30e-006
n = 19    x(n) = 1.0986134397098795    error =  1.15e-006
n = 20    x(n) = 1.0986128642689437    error =  5.76e-007
n = 21    x(n) = 1.0986125767230988    error =  2.88e-007
n = 22    x(n) = 1.0986124332994223    error =  1.45e-007
n = 23    x(n) = 1.0986123606562614    error =  7.20e-008
n = 24    x(n) = 1.0986123234033585    error =  3.47e-008
n = 25    x(n) = 1.0986123085021973    error =  1.98e-008
n = 26    x(n) = 1.0986122936010361    error =  4.93e-009
n = 27    x(n) = 1.0986122786998749    error = -9.97e-009
n = 28    x(n) = 1.0986122488975525    error = -3.98e-008
n = 29    x(n) = 1.0986121892929077    error = -9.94e-008
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n = 30    x(n) = 1.0986120700836182    error = -2.19e-007
n = 31    x(n) = 1.0986118316650391    error = -4.57e-007
n = 32    x(n) = 1.0986118316650391    error = -4.57e-007
n = 33    x(n) = 1.0986118316650391    error = -4.57e-007
n = 34    x(n) = 1.0986099243164063    error = -2.36e-006
n = 35    x(n) = 1.0986099243164063    error = -2.36e-006
n = 36    x(n) = 1.0986022949218750    error = -9.99e-006
n = 37    x(n) = 1.0986022949218750    error = -9.99e-006
n = 38    x(n) = 1.0985717773437500    error = -4.05e-005
n = 39    x(n) = 1.0985107421875000    error = -1.02e-004
n = 40    x(n) = 1.0983886718750000    error = -2.24e-004
n = 41    x(n) = 1.0981445312500000    error = -4.68e-004
n = 42    x(n) = 1.0976562500000000    error = -9.56e-004
n = 43    x(n) = 1.0976562500000000    error = -9.56e-004
n = 44    x(n) = 1.0976562500000000    error = -9.56e-004
n = 45    x(n) = 1.0937500000000000    error = -4.86e-003
n = 46    x(n) = 1.0937500000000000    error = -4.86e-003
n = 47    x(n) = 1.0937500000000000    error = -4.86e-003
n = 48    x(n) = 1.0625000000000000    error = -3.61e-002
n = 49    x(n) = 1.0000000000000000    error = -9.86e-002
n = 50    x(n) = 1.0000000000000000    error = -9.86e-002
n = 51    x(n) = 1.0000000000000000    error = -9.86e-002
n = 52    x(n) = 1.0000000000000000    error = -9.86e-002
n = 53    x(n) = 0.0000000000000000    error = -1.10e+000
n = 54    x(n) = 0.0000000000000000    error = -1.10e+000
n = 55    x(n) = 0.0000000000000000    error = -1.10e+000
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The exact result is $\ln(3) \approx 1.09861228866811$. Notice that the error decreases with each iteration until the 26th iteration when the error is about 5×10^{-9}. After that the error changes sign and begins to increase in magnitude. When $x_n/2^n = \epsilon_{\text{mach}} = 2^{-53}$, $1 + x_n/2^n$ is rounded to give 1, and so $x_{n+1} = 0$. Since x_n is approximately given by 1, this happens when $n = 53$. All subsequent iterations also give 0. This is exactly what we find.

Thus the closest we get to the correct answer is term 26, which gives an error of about 5×10^{-9}. In order to do better, we need to eliminate the problem with cancellation. This can be accomplished by rationalization as we did for the quadratic equation formula. We also want to make the iteration more efficient by letting $y_n = x_n/2^n$, and so $y_0 = x_0$. The recursion relation is then given by

$$y_{n+1} = \sqrt{1 + y_n} - 1.$$

We now rationalize the expression

$$y_{n+1} = \left(\sqrt{1+y_n} - 1\right) \frac{\sqrt{1+y_n}+1}{\sqrt{1+y_n}+1} = \frac{y_n}{\sqrt{1+y_n}+1}.$$

This expression is now free of cancellation error. Iteration using this expression gives the result

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n =  1     x(n) = 1.4641016151377546     error =  3.65e-001
n =  2     x(n) = 1.2642960518099700     error =  1.66e-001
n =  3     x(n) = 1.1776215235190168     error =  7.90e-002
n =  4     x(n) = 1.1372077291666316     error =  3.86e-002
n =  5     x(n) = 1.1176885465556672     error =  1.91e-002
n =  6     x(n) = 1.1080957635770929     error =  9.48e-003
n =  7     x(n) = 1.1033404504923559     error =  4.73e-003
n =  8     x(n) = 1.1009729865915661     error =  2.36e-003
n =  9     x(n) = 1.0997917932434356     error =  1.18e-003
n = 10     x(n) = 1.0992018300290123     error =  5.90e-004
n = 11     x(n) = 1.0989070066380850     error =  2.95e-004
n = 12     x(n) = 1.0987596344781292     error =  1.47e-004
n = 13     x(n) = 1.0986859582797086     error =  7.37e-005
n = 14     x(n) = 1.0986491226505979     error =  3.68e-005
n = 15     x(n) = 1.0986307054535311     error =  1.84e-005
n = 16     x(n) = 1.0986214970093653     error =  9.21e-006
n = 17     x(n) = 1.0986168928258737     error =  4.60e-006
n = 18     x(n) = 1.0986145907437759     error =  2.30e-006
n = 19     x(n) = 1.0986134397051390     error =  1.15e-006
n = 20     x(n) = 1.0986128641864232     error =  5.76e-007
n = 21     x(n) = 1.0986125764272161     error =  2.88e-007
n = 22     x(n) = 1.0986124325476505     error =  1.44e-007
n = 23     x(n) = 1.0986123606078770     error =  7.19e-008
n = 24     x(n) = 1.0986123246379926     error =  3.60e-008
n = 25     x(n) = 1.0986123066530511     error =  1.80e-008
n = 26     x(n) = 1.0986122976605803     error =  8.99e-009
n = 27     x(n) = 1.0986122931643449     error =  4.50e-009
n = 28     x(n) = 1.0986122909162273     error =  2.25e-009
n = 29     x(n) = 1.0986122897921684     error =  1.12e-009
n = 30     x(n) = 1.0986122892301391     error =  5.62e-010
n = 31     x(n) = 1.0986122889491245     error =  2.81e-010
n = 32     x(n) = 1.0986122888086172     error =  1.41e-010
n = 33     x(n) = 1.0986122887383636     error =  7.03e-011
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n = 34    x(n) = 1.0986122887032368    error =  3.51e-011
n = 35    x(n) = 1.0986122886856735    error =  1.76e-011
n = 36    x(n) = 1.0986122886768919    error =  8.78e-012
n = 37    x(n) = 1.0986122886725009    error =  4.39e-012
n = 38    x(n) = 1.0986122886703054    error =  2.20e-012
n = 39    x(n) = 1.0986122886692076    error =  1.10e-012
n = 40    x(n) = 1.0986122886686587    error =  5.49e-013
n = 41    x(n) = 1.0986122886683842    error =  2.75e-013
n = 42    x(n) = 1.0986122886682472    error =  1.38e-013
n = 43    x(n) = 1.0986122886681786    error =  6.91e-014
n = 44    x(n) = 1.0986122886681444    error =  3.49e-014
n = 45    x(n) = 1.0986122886681273    error =  1.78e-014
n = 46    x(n) = 1.0986122886681189    error =  9.33e-015
n = 47    x(n) = 1.0986122886681144    error =  4.88e-015
n = 48    x(n) = 1.0986122886681124    error =  2.89e-015
n = 49    x(n) = 1.0986122886681116    error =  2.00e-015
n = 50    x(n) = 1.0986122886681111    error =  1.55e-015
n = 51    x(n) = 1.0986122886681109    error =  1.33e-015
n = 52    x(n) = 1.0986122886681109    error =  1.33e-015
n = 53    x(n) = 1.0986122886681109    error =  1.33e-015
n = 54    x(n) = 1.0986122886681109    error =  1.33e-015
n = 55    x(n) = 1.0986122886681109    error =  1.33e-015
```

The error is seen to monotonically decrease to a value a few times the machine epsilon. Although there is no cancellation error, rounding error has limited the computation to this accuracy. Although x_n remains unchanged for high n, y_n keeps on decreasing. When $y_n = x_n/2^n \approx \epsilon_{\text{mach}} = 2^{-53}$, and, as we saw before, this occurs when $n = 53$, we have $fl(1 + y_n) = 1$ and so the square root of 1 is 1, and consequently the denominator has the value of 2. Therefore the iteration gives $y_{n+1} = y_n/2$. In terms of x_n that means that $x_{n+1} = 2^{n+1}y_{n+1} = 2^{n+1}y_n/2 = 2^n y_n = x_n$. Thus the resulting value of x_n does not change. This is clearly seen from the result of the computation. So after the iteration gives the correct result for x_n to within a few times the machine epsilon, further iteration does not change that result, unlike the iteration using the original sequence.