POLYTECHNIC UNIVERSITY
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# A Wacky Graph of a Simple Looking Function: The Problem

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**Abstract:** Plotting of a seemingly simple function can sometimes give unexpected wacky results that are due to the finiteness of the underlying floating-point system.

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Suppose we are given a function

$$f(x) = \frac{1 - \cos(x)}{x^2},$$

and we want to know how it looks like by plotting it for specific ranges of values for $x$. One very easy way is to use the built-in function called `ezplot` in MATLAB. This can be done by typing the following at the command prompt

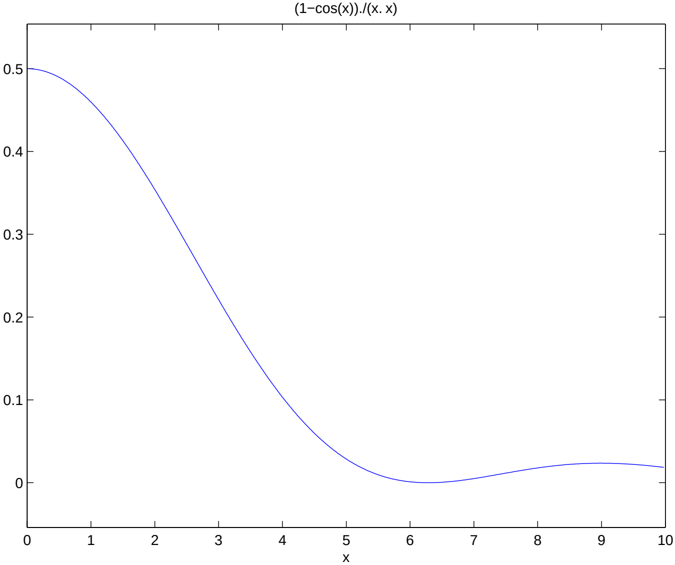
```
ezplot('(1-cos(x))/(x*x)',[0,10]);
```

The function was specified by the first argument of the `ezplot` function, and the second argument gives the range of $x$ values for the plot. The plot looks great and it was certainly very easy to make indeed.

Suppose we are more interested in the range where $x$ is small. We can shorten the range and plot again by entering the command

```
ezplot('(1-cos(x))/(x*x)',[0,1e-2]);
```

Here the number `1e-2` means $1 \times 10^{-2}$. The resulting graph looks rather uninteresting. However on closer inspection we notice a small blip in the curve near $x = 0$. Actually MATLAB's `ezplot` function does not plot the end points of the given interval. It seems that the


(1−cos(x))./(x. x)
0.5
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7 8 9 10
x

blip cannot be due to the potential singularity of the function at $x = 0$. One can do a closer inspection of the blip by using the zoom feature in MATLAB's graphics window.

Re-plotting the function for an even smaller range of $x$ values gives even wackier results. We want to find out what is really going on.

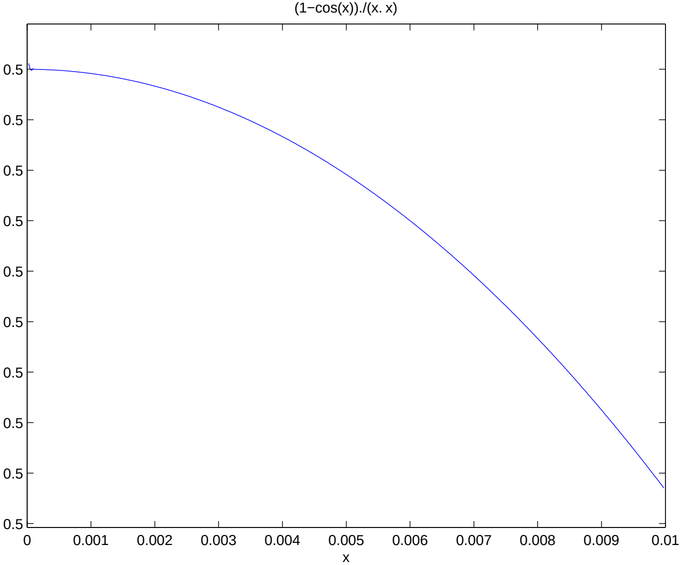
First let us use calculus to find out the proper behavior of the function for $x$ at the origin. When $x = 0$ both the numerator and denominator of the function go to zero, so we need to use the l'Hôpital's rule to get

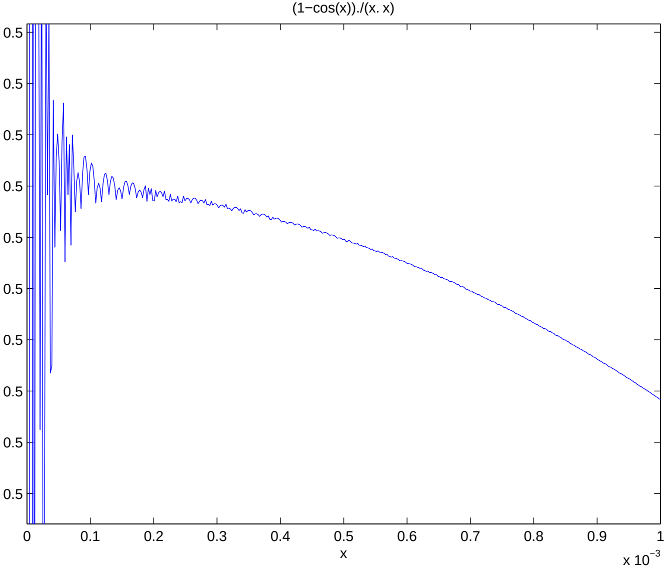
$$\lim_{x\to 0} f(x) = \frac{\frac{d}{dx}(1-\cos x)|_{x=0}}{\frac{d}{dx}(x^2)|_{x=0}} = \frac{\sin x|_{x=0}}{2x|_{x=0}}.$$

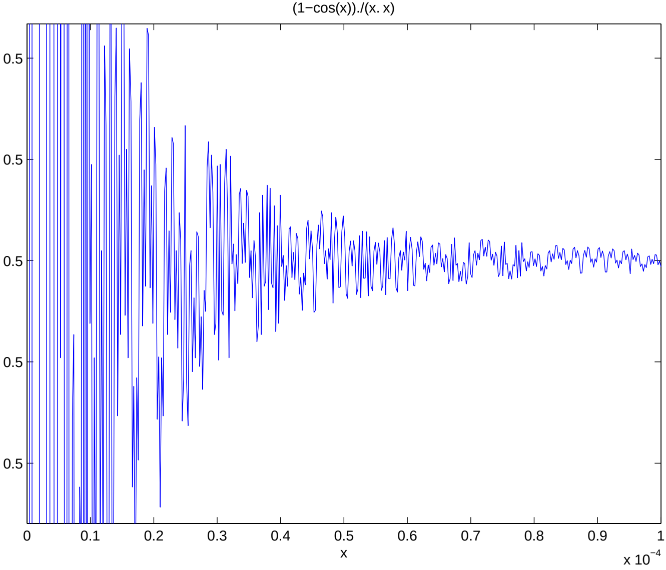
The numerator and denominator still both go to zero at $x = 0$, so we have use the l'Hôpital's rule again to get

$$\lim_{x\to 0} f(x) = \frac{\frac{d}{dx}\sin x|_{x=0}}{\frac{d}{dx}2x|_{x=0}} = \frac{\cos x|_{x=0}}{2|_{x=0}} = \frac{1}{2}.$$

Therefore we see that the function must approach the constant value of $1/2$ at $x = 0$. Thus the results we are seeing here are highly


(1−cos(x))./(x. x)
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0
0.001
0.002
0.003
0.004
0.005
0.006
0.007
0.008
0.009
0.01
x

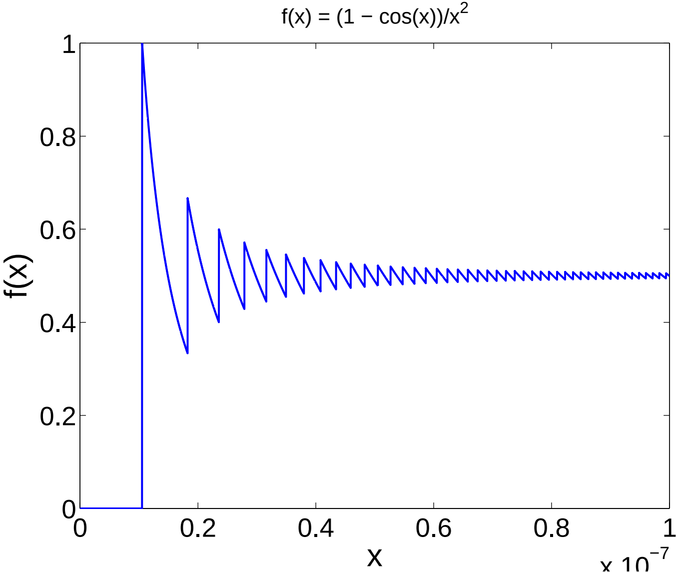

(1−cos(x))./(x. x)
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
x $10^{-3}$


(1−cos(x))./(x. x)
0.5
0.5
0.5
0.5
0.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
x 10−4

incorrect.

To try to understand the cause of the problem we need to write a program to compute the function and analyze the results ourselves. We concentrate on the interval $[0, 1 \times 10^{-7}]$ and obtain the result as shown in the following graph. The MATLAB program, `wackyPlot0.m` is available on the course website. This result is clearly also incorrect.

The graph has a lot of prominent features and looks rather complicated. When we understand how real numbers are represented in a digital computer, it turns out that we will be able to understand quantitatively every single feature exhibited in this graph.


f(x) = (1 − cos(x))/x$^2$
f(x)
x
x 10$^{-7}$
0
0.2
0.4
0.6
0.8
1
0
0.2
0.4
0.6
0.8
1