Direct Derivation of the Sherman-Morrison Formula

The Sherman-Morrison formula can be derived directly by solving the linear problem

$$\left(\mathbf{A} - \mathbf{u}\mathbf{v}^T\right)\mathbf{x} = \mathbf{b}$$

for $\mathbf{x}$ assuming $\mathbf{A}^{-1}$ is already known. To begin we pre-multiply this equation by $\mathbf{A}^{-1}$ and denote $\mathbf{A}^{-1}\mathbf{u} = \mathbf{z}$ and $\mathbf{A}^{-1}\mathbf{b} = \mathbf{y}$ to give

$$\mathbf{x} - \mathbf{z}\mathbf{v}^T\mathbf{x} = \mathbf{y}.$$

Notice that $\mathbf{v}^T\mathbf{x}$ is a scalar quantity, which we denote by α. To find α, we pre-multiply the above equation by $\mathbf{v}^T$ to give

$$\alpha - \mathbf{v}^T\mathbf{z}\alpha = \mathbf{v}^T\mathbf{y}.$$

Since $\mathbf{v}^T\mathbf{z}$ and $\mathbf{v}^T\mathbf{y}$ in the above equation are scalars, we can easily solve for α to get

$$\alpha = \frac{\mathbf{v}^T\mathbf{y}}{1 - \mathbf{v}^T\mathbf{z}}.$$

Thus the solution can be written as

$$\mathbf{x} = \mathbf{y} + \alpha\mathbf{z} \qquad (1)$$

$$
\begin{aligned}
&= \mathbf{A}^{-1}\mathbf{b} + \mathbf{A}^{-1}\mathbf{u}(1 - \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u})^{-1}\mathbf{v}^T\mathbf{A}^{-1}\mathbf{b} \\
&= \left[\mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{u}(1 - \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u})^{-1}\mathbf{v}^T\mathbf{A}^{-1}\right]\mathbf{b}.
\end{aligned}
$$

We then obtain from this equation the Sherman-Morrison formula

$$
(\mathbf{A} - \mathbf{u}\mathbf{v}^T)^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{u}(1 - \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u})^{-1}\mathbf{v}^T\mathbf{A}^{-1}.
$$