Divided-Differences

Given a set of data points $(t_1, y_1), \cdots, (t_N, y_N)$, an alternate method for computing the coefficients x_j of the Newton polynomial interpolant is via quantities known as divided differences, which are usually denoted by $f[\,]$ and are defined recursively by the formula

$$f[t_1, t_2, \cdots, t_k] = \frac{f[t_2, t_3, \cdots, t_k] - f[t_1, t_2, \cdots, t_{k-1}]}{t_k - t_1}$$

for $k = 2, \cdots$. The recursion starts with $f[t_i] = y_i, i = 1, \cdots, N$. The parameters t_1, t_2, $\cdots$, should be treated like independent variables. It turns out that the coefficient of the jth basis function in the Newton interpolant is given by $x_j = f[t_1, t_2, \cdots, t_j]$. Like forward substitution, use of this recursion requires only $\mathcal{O}(n^2)$ arithmetic operations to compute the coefficients, but it is less prone to overflow or underflow than is direct formation of the entries of the triangular Newton basis matrix.

We illustrate the method of divided differences for the case where $N = 4$.

$$\begin{array}{llll}
f[t_1] = y_1 & & & \\
f[t_2] = y_2 & f[t_1, t_2] = \dfrac{f[t_2] - f[t_1]}{t_2 - t_1} & & \\
f[t_3] = y_3 & f[t_2, t_3] = \dfrac{f[t_3] - f[t_2]}{t_3 - t_2} & f[t_1, t_2, t_3] = \dfrac{f[t_2, t_3] - f[t_1, t_2]}{t_3 - t_1} & \\
f[t_4] = y_4 & f[t_3, t_4] = \dfrac{f[t_4] - f[t_3]}{t_4 - t_3} & f[t_2, t_3, t_4] = \dfrac{f[t_3, t_4] - f[t_2, t_3]}{t_4 - t_2} & f[t_1, t_2, t_3, t_4] = \dfrac{f[t_2, t_3, t_4] - f[t_1, t_2, t_3]}{t_4 - t_1}
\end{array}$$