EXAMPLE

Your favorite aunt is loaning you $10,000 for one year without interest to start your investing career. At the end of the year you must return the $10,000 or she will cut you out of her will which you definitely do not want. You are determined (with probability 1) that this should not happen, but you want to maximize your expected return. You are not allowed to borrow more money to invest or to sell short investments. Since this is an academic discussion, you don't have to pay transaction costs either, and you can buy any of the investments in any amount consistent with the other constraints.

You have several options. You can buy a T-Bill with one year maturity which pays 7%, or you can buy one of three well diversified mutual funds, an index fund, a utilities fund, or a growth fund. Their return in one year depends on whether the market (= the index fund) has a "good," "normal" or "down" year. The probability of each of these results is 0.3, 0.4, and 0.3, respectively. The returns for each fund in each kind of year is given in Table 1, along with the expected return for each kind of fund averaged over the type of year. For example, to compute the expected return for the growth fund we have $E = 0.3 \times (-0.10) + 0.4 \times 0.13 + 0.3 \times 0.20 = 0.082$.

<table>
<thead>
<tr>
<th></th>
<th>Down Yr.</th>
<th>Normal Yr.</th>
<th>Good Yr.</th>
<th>Expected Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.3</td>
<td>0.4</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>T-Bill</td>
<td>0.07</td>
<td>0.07</td>
<td>0.07</td>
<td>0.07</td>
</tr>
<tr>
<td>Growth Fund</td>
<td>-0.10</td>
<td>0.13</td>
<td>0.2</td>
<td>0.082</td>
</tr>
<tr>
<td>Index Fund</td>
<td>-0.04</td>
<td>0.11</td>
<td>0.16</td>
<td>0.08</td>
</tr>
<tr>
<td>Utility Fund</td>
<td>-0.02</td>
<td>0.10</td>
<td>0.145</td>
<td>0.0775</td>
</tr>
</tbody>
</table>

Now we represent our problem in mathematical terms. Before we do this we are going to renormalize our $10,000 to one unit. That is our unit will be worth $10,000. Now we let $x_T$ be the fraction of our money that we invest in treasury bills; similarly, $x_G$, $x_I$ and $x_U$ represent the amount invested in the growth, index and utility funds.

We first note that in our new units, $x_T + x_G + x_I + x_U = 1$, and $x_T \geq 0$, $x_G \geq 0$, $x_I \geq 0$, $x_U \geq 0$. These relations represent the fact that we cannot invest more money than we have, and that we cannot short any of the investments.

Next we must guarantee that we always, at least, break even. It is obvious from the data that if we break even in a down year we will do all right in normal or good years. The following relation guarantees that we break even in down years.

$$0.07x_T - 0.10x_G - 0.04x_I - 0.02x_U \geq 0.$$  

Finally, we construct the function which represents the expected return that we are attempting to maximize:
ER = 0.07x_T + 0.082x_G + 0.08x_I + 0.0775x_U

To summarize, we seek to choose \( x_T, x_G, x_I \) and \( x_U \) to maximize the expected return, \( ER \), subject to the following relations:

Maximize \( ER = 0.07x_T + 0.082x_G + 0.08x_I + 0.0775x_U \)

Subject to
\[
\begin{align*}
0.07x_T + 0.10x_G + 0.04x_I - 0.02x_U & \geq 0 \\
x_T & \geq 0, \quad x_G \geq 0, \quad x_I \geq 0, \quad x_U \geq 0.
\end{align*}
\]

This kind of problem is called a linear program. It is an optimization problem that is characterized by a linear objective function which is to be minimized or maximized and linear constraints which may be equations or inequalities. We will look at the solution of general problems of this type in the next few sections, but this one is simple enough that we can solve it directly, hopefully gaining some insight as we do so.

In order to represent the problem graphically we interpret the \( x \)'s as weights in an averaging process. This is reasonable because they are non-negative and sum to one. We will represent the coefficients of each variable as a point in euclidean space with horizontal coordinate equal to the return in a down year and the vertical coordinate as the expected return. See Figure 1 where the four different investments are represented.

By weighting the investments with the \( x \)'s we can get \(<\text{expected return, down year return}>\) payoffs for portfolios anywhere in the shaded region. In order to satisfy the requirement that we always get at least 0 return we must be in the right part of the diagram as indicated by the arrows in Figure 1. Finally, maximizing the expected return on our portfolio means finding the highest point that is in the shaded figure and satisfies the 0 return constraint; i.e., we want the highest point in the black area in the figure. The optimal solution is indicated.

From the figure we see that the optimum point is on the line connecting the treasury point to the index fund. Therefore the optimum strategy is to invest in T-bills and the index fund in the appropriate combination so that the return in down years is exactly 0. It turns out that you will invest $10,000 \times (0.3636) = $3,636 in T-bills and $10,000 \times (0.6364) = $6,364 in the index fund for an expected return of $10,000 \times 0.0764 = $764.

Exercises:

1. Perform the computations to confirm the values given above for the amounts invested and the expected return for the optimum portfolio.

2. The point corresponding to the utilities fund is in the interior of the shaded area of Figure 1. What does this imply about the utility fund?

3. Suppose that instead of requiring a return of 0 in down years we allow a loss of 7%. What investments would be represented in the optimal portfolio now?