DUALITY

Dual Problems:

Here it is convenient to use the symmetric linear program in matrix form which we shall call the **primal problem** for reasons that will become clear shortly:

\[ (P) \quad z = \text{minimum } cx \]
subject to \[ Ax \geq b \]
and \[ x \geq 0 \]

Associated with (P) is its **dual**, (D), given by:

\[ (D) \quad w = \text{maximum } yb \]
subject to \[ yA \leq c \]
and \[ y \geq 0. \]

Note that \( x \) is a column vector with \( n \) components, \( y \) is a row vector with \( m \) components, \( b \) is a column vector with \( m \) components, \( c \) is a row vector with \( n \) components, and \( A \) has \( m \) rows and \( n \) columns. We will say that a vector \( x \) is **feasible** for (P) if it satisfies the constraints; it may or may not minimize the objective. Similarly, \( y \) is feasible in (D) if it satisfies the constraints of (D). We will often use \( x^* \) and \( y^* \) to denote optimal solutions to (P) and (D), respectively. The two problems (P) and (D) are intimately related. They are called **duals** of one another. We will prove a theorem called the duality theorem which pins this down shortly; however, we will want to use the simplex method to prove it which we haven't introduced as yet. For now we are content with the:

**Weak Duality:**

**Theorem (Weak Duality):** For any \( x \) feasible in (P) and \( y \) feasible in (D) we have \( cx \geq yb \).

Proof: Consider \( yAx \). We have \( yAx \geq yb \) from the constraint of (P), and we have \( yAx \leq cx \) from the constraint of (D). Thus \( cx \geq yAx \geq yb \).

This has immediate, important consequences. Let \( x^* \) denote an optimal solution to (P) and let \( y^* \) denote an optimal solution to (D) we have:

**Corollary 1:** \( cx^* \geq yb \) for any feasible \( y \); that is, \( yb \), for any \( y \) is a lower bound on the optimal solution of (P).

Similarly.

**Corollary 2:** \( y^*b \leq cx \) for any feasible \( x \).

Finally.

**Corollary 3:** \( y^*b \leq cx^* \).

Of course we are implicitly assuming that optimal solutions to (P) and/or (D) exist when we write \( x^* \) and/or \( y^* \), respectively.
Note, that if \( yb = cx \) for any feasible \( x \) and \( y \), then both must be optimal for their respective problems. For example, by the theorem \( yb \leq cx \), for any \( y' \). That is, \( cx \) is an upper bound for \( (D) \). But \( yb \) equals the upper bound so that no \( y' \) can do better; therefore, \( y \) is optimal. A similar argument holds for \( x \). The strong duality theorem will establish that equality holds if both problems are feasible.

**Corollary 4:** If \( cx = yb \) for any \( x \) and \( y \), feasible for \( (P) \) and \( (D) \), respectively, then \( x \) and \( y \) are both optimal.

**Complementary Slackness:**

In order to gain a little facility with matrix and vector notation, let us assume, for the time being, that equality is possible. What are conditions for this to be so. From the theorem we see that we must have \( cx^* = y^*Ax^* = by^* \) for optimal solutions \( x^* \) and \( y^* \). From the first equality we have that \( (c-y^*A)x^* = 0 \). To explore further the implications of this, let us expand the vector/matrix representation using summations:

\[
(c-y^*A)x^* = \sum_{j=1}^{n} (c_j - y^*A^j)x_j^*
\]

Note that the product in each term of the sum is the product of two non-negative values, so that each term is non-negative, so that each term separately has to be zero. In order for this to happen, one or both of the factors must be zero. Similarly, the implication of \( yAx = by \) is that \( y(Ax-b) = 0 \), and:

\[
y^*(Ax^* - b) = \sum_{i=1}^{m} y^*_i (A^i x^* - b_i)
\]

We can summarize all this by:

**Theorem (Complementary Slackness):** Let \( x \) and \( y \) be feasible for \( (P) \) and \( (D) \) respectively, then \( cx = yb \) if and only if:

For each \( i, i=1,...,m \) we have \( y_i = 0 \) or \( A_i x = b_i \),

and

For each \( j, j=1,...,n \) we have \( x_j = 0 \) or \( yA^j = c_j \).