Direct Derivation of the Sherman-Morrison Formula

The Sherman-Morrison formula can be derived directly by solving the linear problem

$$(A - uv^T)x = b$$

for $x$ assuming $A^{-1}$ is already known. To begin we pre-multiply this equation by $A^{-1}$ and denote $A^{-1}u = z$ and $A^{-1}b = y$ to give

$$x - zv^Tx = y.$$
Notice that $\mathbf{v}^T \mathbf{x}$ is a scalar quantity, which we denote by $\alpha$. To find $\alpha$, we pre-multiply the above equation by $\mathbf{v}^T$ to give

$$\alpha - \mathbf{v}^T \mathbf{z} \alpha = \mathbf{v}^T \mathbf{y}.$$ 

Since $\mathbf{v}^T \mathbf{z}$ and $\mathbf{v}^T \mathbf{y}$ in the above equation are scalars, we can easily solve for $\alpha$ to get

$$\alpha = \frac{\mathbf{v}^T \mathbf{y}}{1 - \mathbf{v}^T \mathbf{z}}.$$ 

Thus the solution can be written as

$$\mathbf{x} = \mathbf{y} + \alpha \mathbf{z} \quad (1)$$
\[
= A^{-1}b + A^{-1}u(1 - v^T A^{-1}u)^{-1}v^T A^{-1}b \\
= \left[A^{-1} + A^{-1}u(1 - v^T A^{-1}u)^{-1}v^T A^{-1}\right]b.
\]

We then obtain from this equation the Sherman-Morrison formula

\[
(A - uv^T)^{-1} = A^{-1} + A^{-1}u(1 - v^T A^{-1}u)^{-1}v^T A^{-1}.
\]