A Wacky Graph of a Simple Looking Function: The Problem

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Abstract: Plotting of a seemingly simple function can sometimes give unexpected wacky results that are due to the finiteness of the underlying floating-point system.
Suppose we are given a function

\[ f(x) = \frac{1 - \cos(x)}{x^2}, \]

and we want to know how it looks like by plotting it for specific ranges of values for \( x \). One very easy way is to use the built-in function called `ezplot` in MATLAB. This can be done by typing the following at the command prompt:

```matlab
ezplot('(1-cos(x))./(x.*x)',[0,10]);
```

The function was specified by the first argument of the `ezplot` function, and the second argument gives the range of \( x \) values for the plot. The plot looks great and it was certainly very easy to make indeed.

Suppose we are more interested in the range where \( x \) is small. We can shorten the range and plot again by entering the command:

```matlab
ezplot('(1-cos(x))./(x.*x)',[0,1e-2]);
```

Here the number \( 1e-2 \) means \( 1 \times 10^{-2} \). The resulting graph looks rather uninteresting. However on closer inspection we notice a small blip in the curve near \( x = 0 \). Actually MATLAB’s `ezplot` function does not plot the end points of the given interval. It seems that the
\[
\frac{(1-\cos(x))}{(x \cdot x)}
\]
blip cannot be due to the potential singularity of the function at $x = 0$. One can do a closer inspection of the blip by using the zoom feature in MATLAB’s graphics window.

Re-plotting the function for an even small range of $x$ values gives even wackier results. We want to find out what is really going on.
\[(1 - \cos(x)) / (x \cdot x)\]
\[ \frac{(1-\cos(x))}{x \cdot x} \]
First let us use calculus to find out the proper behavior of the function for $x$ at the origin. When $x = 0$ both the numerator and denominator of the function go to zero, so we need to use the l’Hôpital’s rule to get

$$\lim_{x \to 0} f(x) = \frac{\frac{d}{dx} (1 - \cos x)|_{x=0}}{\frac{d}{dx} (x^2)|_{x=0}} = \frac{\sin x|_{x=0}}{2x|_{x=0}}.$$  

The numerator and denominator still both go to zero at $x = 0$, so we have use the l’Hôpital’s rule again to get

$$\lim_{x \to 0} f(x) = \frac{\frac{d}{dx} \sin x|_{x=0}}{\frac{d}{dx} 2x|_{x=0}} = \frac{\cos x|_{x=0}}{2|_{x=0}} = \frac{1}{2}.$$  

Therefore we see that the function must approach the constant value of $1/2$ at $x = 0$. Thus the results we are seeing here are highly incorrect.

To try to understand the cause of the problem we need to write a program to compute the function and analyze the results ourselves. We concentrate on the interval $[0, 1 \times 10^{-7}]$ and obtain the result as shown in the following graph. The MATLAB program, `wackyPlot0.m`
is available on the course website. This result is clearly also incorrect.

The graph has a lot of prominent features and looks rather complicated. When we understand how real numbers are represented in a digital computer it turns out that we will be able to understand quantitatively every single feature exhibited in this graph.